Supplemental Materials

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This document includes the supplemental materials for the paper titled "Performance Analysis of Real-Time Detection in Fusion-based Sensor Networks."

Appendix A
Target Localization Performance

In each detection period, a sensor participates in the target localization if its reading exceeds a threshold ζ. Let (X_i, Y_i) denote the coordinates of sensor i and suppose there are m sensors participating in the localization. The target is localized at the geometric center of these sensors, i.e., $\bar{X} = \frac{1}{m}\sum_{i=1}^{m} X_i$ and $\bar{Y} = \frac{1}{m}\sum_{i=1}^{m} Y_i$. Let (d_i, θ_i) denote the coordinates of sensor i in the polar coordinate plane with origin at the target. Due to the Poisson process, θ_i is uniformly distributed in $(0, 2\pi)$. It is easy to verify that $(\bar{X}, \bar{Y})$ are the unbiased estimator of the target position. Hence, the mean square error (MSE) of $\bar{X}$ is $\mathrm{MSE}(\bar{X}) = \mathrm{Var}[\bar{X}] = \frac{1}{m}\mathrm{Var}[X_i] = \frac{1}{2m}\mathbb{E}[d_i^2]$. We now prove that $\mathbb{E}[d_i^2]$ is upper-bounded. As $y_i = S \cdot w(d_i) + n_i \geq \zeta$, $d_i \leq w^{-1}\left(\frac{\zeta - n_i}{S}\right)$. Hence, $\mathbb{E}[d_i^2] \leq \mathbb{E}\left[\left(w^{-1}\left(\frac{\zeta-n_i}{S}\right)\right)^2\right]$. As a result, $\mathrm{MSE}(\bar{X}) = \mathcal{O}\left(\frac{1}{m}\right)$. Note that m increases statistically with the network density.

Appendix B
Deriving the Mean And Variance of s_i

We first prove that $\{s_i | i \in \mathbf{F}_j\}$ are independent and identically distributed (*i.i.d.*) for any target position P. As sensors are deployed uniformly and independently, $\{d_i | i \in \mathbf{F}_j\}$ are *i.i.d.* for any P, where d_i is the distance between sensor i and point P. Therefore, $\{s_i | i \in \mathbf{F}_j\}$ are *i.i.d.* for any P, as s_i is a function of d_i, i.e., $s_i = S \cdot w(d_i)$.

We then derive the mean and variance of s_i, i.e., μ_s and σ_s^2. Let (x_p, y_p) and (x_i, y_i) denote the coordinates of point P and sensor i, respectively. The posterior probability density function (PDF) of (x_i, y_i) is $f(x_i, y_i) = \frac{1}{\pi R^2}$ where $(x_i - x_p)^2 + (y_i - y_p)^2 \leq R^2$. Hence, the posterior cumulative distribution function (CDF) of d_i is given by $F(d_i) = \int_0^{2\pi} \mathrm{d}\theta \int_0^{d_i} \frac{1}{\pi R^2} \cdot x \mathrm{d}x = \frac{d_i^2}{R^2}$, where $d_i \in [0, R]$. Therefore,

$$\mu_s = \int_0^R S w(d_i) \mathrm{d}F(d_i) = \frac{2S}{R^2}\int_0^R w(d_i) d_i \mathrm{d}d_i,$$

$$\sigma_s^2 = \int_0^R S^2 w^2(d_i) \mathrm{d}F(d_i) - \mu_s^2 = \frac{2S^2}{R^2}\int_0^R w^2(d_i) d_i \mathrm{d}d_i - \mu_s^2.$$

By letting $\mu_0 = \frac{2}{R^2}\int_0^R w(d_i) d_i \mathrm{d}d_i$ and $\sigma_0^2 = \frac{2}{R^2}\int_0^R w^2(d_i) d_i \mathrm{d}d_i - \mu_0^2$, we have $\mu_s = S\mu_0$ and $\sigma_s^2 = S^2\sigma_0^2$.

Appendix C
The Proof of Theorem 1

Proof: Denote A_j as the event that the target is not detected in the j^{th} unit detection. Thus, the probability of A_j is $\mathbb{P}(A_j) = 1 - P_{Dj}$. Suppose the target is detected in the J^{th} unit detection. Although the intrusion detection is a series of infinite Bernoulli trials, J does not follow the geometric distribution because the success probability of each Bernoulli trial (i.e., P_{Dj}) is a random variable (RV) rather than a constant. The mean of J is give by

$$\mathbb{E}[J] = 1 \cdot \mathbb{P}(\bar{A}_1) + \sum_{j=2}^{\infty} j \cdot \mathbb{P}\left(\bigcap_{k=1}^{j-1} A_k \bigcap \bar{A}_j\right) \quad (11)$$

$$= 1 - \mathbb{P}(A_1) + \sum_{j=2}^{\infty} j \cdot \left(\mathbb{P}\left(\bigcap_{k=1}^{j-1} A_k\right) - \mathbb{P}\left(\bigcap_{k=1}^{j} A_k\right)\right)$$

$$= 1 + \sum_{j=1}^{\infty} \mathbb{P}\left(\bigcap_{k=1}^{j} A_k\right) \quad (12)$$

$$= 1 + \sum_{j=1}^{\infty}\prod_{k=1}^{j} \mathbb{P}(A_k) \quad (13)$$

$$= 1 + \sum_{j=1}^{\infty}\prod_{k=1}^{j} (1 - P_{Dk}). \quad (14)$$

Note that the $\bigcap_{k=1}^{j-1} A_k \bigcap \bar{A}_j$ in (11) represents the event that the target is not detected from the first to the $(j-1)^{\text{th}}$ unit detection but detected in the j^{th} unit detection. As the measurements in different sampling intervals are mutually independent, $A_j : j \geq 1$ are mutually independent. Hence, Eq. (13) follows. We now explain the physical meaning of $\mathbb{E}[J]$. For a given randomly

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deployed network, if the target always appears at a fixed location and travels a fixed trajectory, according to (4), $\{P_{Dj}|j \geq 1\}$ are fixed values as $\{N_j|j \geq 1\}$ are fixed. As each unit detection is probabilistic, the $\mathbb{E}[J]$ is the average delay of detecting the target with fixed trajectory. For the target that appears at random location and travels arbitrary trajectory, $\{P_{Dj}|j \geq 1\}$ are RVs as $\{N_j|j \geq 1\}$ are RVs. Therefore, the average delay for detecting the target with arbitrary trajectory, i.e., α-delay, is given by $\tau = \mathbb{E}[\mathbb{E}[J]]$, where $\mathbb{E}[\mathbb{E}[J]]$ is the average of $\mathbb{E}[J]$ taken over all possible target trajectories. As discussed in Section 5.2, if fusion ranges do not overlap, $\{N_j|j \geq 1\}$ are *i.i.d.* RVs. Hence, $\{P_{Dj}|j \geq 1\}$ are also *i.i.d.* RVs. Therefore, $\tau = \mathbb{E}[\mathbb{E}[J]] = 1 + \sum_{j=1}^{\infty} \prod_{k=1}^{j} \mathbb{E}[1 - P_{Dk}] = 1 + \sum_{j=1}^{\infty}(1 - \mathbb{E}[P_D])^j = \frac{1}{\mathbb{E}[P_D]}$. □

Appendix D
Proof of Lemma 2

Proof: We abuse the symbols a bit to use N instead of N_j and P_D instead of P_{Dj} as we are not interested in the index of unit detection. As $\rho \to \infty$, $N \to \infty$ almost surely. In (4), the second item $-\frac{\mu_s}{\sqrt{\sigma_s^2+\sigma^2}} \cdot \sqrt{N}$ dominates when $\rho \to \infty$, since the first item $\frac{\sigma}{\sqrt{\sigma_s^2+\sigma^2}} \cdot Q^{-1}(\alpha)$ is a constant. Therefore, it's safe to use $P_D = Q(\gamma\sqrt{N})$ to approximate (4), where $\gamma = -\frac{\mu_s}{\sqrt{\sigma_s^2+\sigma^2}}$. From Lemma 1 and Theorem 1, if the same α-delay of τ is achieved under the two models, we have

$$\mathbb{E}[P_D] = 1 - \mathrm{e}^{-\rho_d \pi r^2}. \quad (15)$$

We first prove the lower bound in (6). It is easy to verify that $P_D = Q(\gamma\sqrt{N})$ is a concave function. According to Jensen's inequality, we have $\mathbb{E}[P_D] \leq Q(\gamma\sqrt{\mathbb{E}[N]}) = Q(\gamma\sqrt{\rho_f \pi R^2})$. From (15), we have $1 - \mathrm{e}^{-\rho_d \pi r^2} = \mathbb{E}[P_D] \leq Q(\gamma\sqrt{\rho_f \pi R^2})$. Accordingly, $\rho_d \leq -\frac{1}{\pi r^2} \ln \Phi(\gamma\sqrt{\pi}R \cdot \sqrt{\rho_f})$, where $\Phi(x) = 1 - Q(x)$. Hence, the density ratio satisfies

$$\lim_{\tau \to 1^+} \frac{\rho_f}{\rho_d} \geq -\pi r^2 \cdot \lim_{\rho_f \to \infty} \frac{\rho_f}{\ln \Phi(\gamma\sqrt{\pi}R \cdot \sqrt{\rho_f})} = \frac{2}{\gamma^2 R^2} \cdot r^2.$$

In the above derivation, we use the equality $\lim_{x \to \infty} \frac{x}{\ln \Phi(\vartheta\sqrt{x})} = -\frac{2}{\vartheta^2}$, which is proved in Appendix H.

We now prove the upper bound in (6). As $P_D > 0$, according to Markov's inequality, for any given number c, we have

$$\mathbb{E}[P_D] \geq c \cdot \mathbb{P}(P_D \geq c). \quad (16)$$

We define ξ and c as follows:

$$\xi = \frac{\gamma^2 + 2 - \sqrt{\gamma^4 + 4\gamma^2}}{2}, \quad c = Q(\gamma\sqrt{\xi \rho_f \pi R^2}). \quad (17)$$

It's easy to verify that $\xi \in (0, 1)$. Therefore,

$$\mathbb{P}(P_D \geq c) = \mathbb{P}\left(Q(\gamma\sqrt{N}) \geq Q(\gamma\sqrt{\xi \rho_f \pi R^2})\right) = \mathbb{P}(N \geq \xi \rho_f \pi R^2).$$

As $N \sim \mathrm{Poi}(\rho_f \pi R^2)$ and the Poisson distribution approaches the normal distribution $\mathcal{N}(\rho_f \pi R^2, \rho_f \pi R^2)$ when $\rho_f \to \infty$, we have

$$\mathbb{P}(P_D \geq c) = Q\left(\frac{\xi \rho_f \pi R^2 - \rho_f \pi R^2}{\sqrt{\rho_f \pi R^2}}\right) = Q\left((\xi - 1)\sqrt{\rho_f \pi R^2}\right).$$

By replacing c and $\mathbb{P}(P_D \geq c)$ in (16), we have

$$\mathbb{E}[P_D] \geq Q\left(\gamma\sqrt{\xi \rho_f \pi R^2}\right) \cdot Q\left((\xi - 1)\sqrt{\rho_f \pi R^2}\right).$$

It is easy to verify that $\gamma\sqrt{\xi} = \xi - 1$. Thus the above inequality reduces to $\mathbb{E}[P_D] \geq Q^2(h\sqrt{\rho_f})$, where $h = \gamma\sqrt{\xi\pi}R$. From (15), we have $1 - \mathrm{e}^{-\rho_d \pi r^2} = \mathbb{E}[P_D] \geq Q^2(h\sqrt{\rho_f})$. Accordingly, $\rho_d \geq -\frac{1}{\pi r^2} \cdot (\ln(1 + Q(h\sqrt{\rho_f})) + \ln \Phi(h\sqrt{\rho_f}))$. Hence, we have

$$\lim_{\tau \to 1^+} \frac{\rho_f}{\rho_d} \leq -\pi r^2 \lim_{\rho_f \to \infty} \frac{\rho_f}{\ln(1 + Q(h\sqrt{\rho_f})) + \ln \Phi(h\sqrt{\rho_f})}$$
$$= -\pi r^2 \lim_{\rho_f \to \infty} \frac{\rho_f}{\ln \Phi(h\sqrt{\rho_f})} = \frac{2}{\xi \gamma^2 R^2} \cdot r^2. \quad (18)$$

Note that $h = \gamma\sqrt{\xi\pi}R < 0$ and $\ln(1 + Q(h\sqrt{\rho_f})) = \ln 2$ when $\rho_f \to \infty$. We also use the equality $\lim_{x \to \infty} \frac{x}{\ln \Phi(\vartheta\sqrt{x})} = -\frac{2}{\vartheta^2}$ that is proved in Appendix H to derive (18). □

Appendix E
Proof of Lemma 3

Proof: Let A_j denote the event that the target is not detected in the j^{th} unit detection and C_j denote the corresponding target disc. Suppose the target is detected in the J^{th} unit detection. Recall (12), we have $\mathbb{E}[J] = 1 + \sum_{j=1}^{\infty} \mathbb{P}\left(\bigcap_{k=1}^{j} A_k\right) = 1 + \sum_{j=1}^{\infty} \prod_{k=1}^{j} \mathbb{P}\left(A_k \middle| \bigcap_{l=1}^{k-1} A_l\right)$. The above derivation follows the definition of conditional probability. Let C denote the common area between the k^{th} target disc and the union of all the previous target discs, i.e., $C = C_k \cap (\bigcup_{l=1}^{k-1} C_l)$. Therefore, $C \geq 0$ and

$$\mathbb{P}\left(A_k \middle| \bigcap_{l=1}^{k-1} A_l\right) = \mathbb{P}\left(\text{there is no sensor in } (C_k - C)\right)$$
$$= \mathrm{e}^{-\rho(\pi r^2 - C)} \geq \mathrm{e}^{-\rho \pi r^2}.$$

Hence, $\tau = \mathbb{E}[J] \geq 1 + \sum_{j=1}^{\infty} \left(\mathrm{e}^{-\rho \pi r^2}\right)^j = \frac{1}{1 - \mathrm{e}^{-\rho \pi r^2}}$. □

Appendix F
Proof of Theorem 3

Proof: We first introduce the generalized Hölder's inequality [1]. *For random variables X_i, $i = 1, \ldots, n$, we have $\mathbb{E}\left[\prod_{i=1}^{n} |X_i|\right] \leq \prod_{i=1}^{n} \left(\mathbb{E}\left[|X_i|^{p_i}\right]\right)^{1/p_i}$ where $p_i > 1$ and $\sum_{i=1}^{n} p_i^{-1} = 1$.* If X_i, $i = 1, \ldots, n$, are identically distributed, by setting $p_i = n$, we have

$$\mathbb{E}\left[\prod_{i=1}^{n} |X_i|\right] \leq \mathbb{E}\left[|X|^n\right], \quad (19)$$

where X can be any X_i. In our problem, $\{N_j|j \geq 1\}$ are identically distributed RVs due to the Poisson process. As P_{Dj} is a function of N_j (given by (4)), $\{P_{Dj}|j \geq 1\}$ are also identically distributed RVs. Recall (14), by applying the inequality (19) , the α-delay of fusion-based detection can be derived as

$$\begin{aligned}\tau = \mathbb{E}[\mathbb{E}[J]] &= 1 + \sum_{j=1}^{\infty} \mathbb{E}\left[\prod_{k=1}^{j}(1-P_{Dk})\right] \\ &\leq 1 + \sum_{j=1}^{\infty} \mathbb{E}[(1-P_D)^j] = \mathbb{E}\left[\frac{1}{P_D}\right].\end{aligned}$$

□

Appendix G
Proof of Theorem 4

Proof: According to Lemma 3 and Theorem 3, we have

$$1/(1-\mathrm{e}^{-\rho_d \pi r^2}) \leq \tau \leq \mathbb{E}\left[1/P_D\right]. \tag{20}$$

We first find a upper bound of $\mathbb{E}\left[1/P_D\right]$. As discussed in Appendix D, it is safe to use $P_D = Q(\gamma\sqrt{N})$ to approximate (4), where $\gamma = -\frac{\mu_s}{\sqrt{\sigma_s^2+\sigma^2}}$. As $N \sim \mathrm{Poi}(\rho_f \pi R^2)$ and the Poisson distribution approaches to the normal distribution $\mathcal{N}(\rho_f \pi R^2, \rho_f \pi R^2)$ when $\rho_f \to \infty$, for any given constant $\xi \in (0,1)$, we have $\mathbb{P}(N \geq \xi\rho_f \pi R^2) = Q\left(\frac{\xi\rho_f\pi R^2 - \rho_f \pi R^2}{\sqrt{\rho_f \pi R^2}}\right) = Q\big((\xi-1)\sqrt{\rho_f \pi R^2}\big)$. When $\rho_f \to \infty$, $\mathbb{P}(N \geq \xi\rho_f\pi R^2) \to 1$, i.e., $N \geq \xi\rho_f \pi R^2$ with high probability (*w.h.p.*). Moreover, as $1/P_D = 1/Q(\gamma N)$ is a decreasing function of N, $\mathbb{E}[1/P_D] \leq 1/Q(\gamma\sqrt{\xi\rho_f\pi R^2})$ *w.h.p.*. Furthermore, according to (20), we have $1/(1-\mathrm{e}^{-\rho_d \pi r^2}) \leq 1/Q(\gamma\sqrt{\xi\rho_f \pi R^2})$ *w.h.p.* when $\rho_f \to \infty$. After manipulation, we have $\rho_d \geq -\frac{1}{\pi r^2}\ln\left(\Phi(\gamma\sqrt{\xi\pi}R\sqrt{\rho_f})\right)$, where $\Phi(x) = 1 - Q(x)$. Hence, we have

$$\lim_{\tau\to 1^+}\frac{\rho_f}{\rho_d} \leq -\pi r^2 \lim_{\rho_f\to\infty}\frac{\rho_f}{\ln\left(\Phi(\gamma\sqrt{\xi\pi}R\sqrt{\rho_f})\right)} = \frac{2}{\gamma^2 \xi R^2}\cdot r^2. \tag{21}$$

In the above derivation, we use the equality $\lim\limits_{x\to\infty}\frac{x}{\ln\Phi(\vartheta\sqrt{x})} = -\frac{2}{\vartheta^2}$ that is proved in Appendix H. Hence, the upper bound of the density ratio is $\lim_{\tau\to 1^+}\rho_f/\rho_d = \mathcal{O}(r^2)$. As $r^2 = \Theta\left(\left(\frac{\delta}{Q^{-1}(\alpha)}\right)^{2/k}\right)$, we have (8). □

Appendix H
A Limit Used in the Proofs of Lemma 2 and Theorem 4

Denote $\phi(x)$ and $\Phi(x)$ as the PDF and CDF the standard normal distribution, respectively, i.e., $\phi(x) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^2/2}$ and $\Phi(x) = \int_{-\infty}^{x}\phi(t)\mathrm{d}t$. Note that $\Phi'(x) = \phi(x)$ and $\phi'(x) = -x\phi(x)$. For constant $\vartheta < 0$, we have

$$\begin{aligned}\lim_{x\to\infty}\frac{x^2}{\ln\Phi(\vartheta x)} &\overset{(*)}{=} \lim_{x\to\infty}\frac{2x}{\frac{1}{\Phi(\vartheta x)}\phi(\vartheta x)\vartheta} = \frac{2}{\vartheta}\lim_{x\to\infty}\frac{\Phi(\vartheta x)x}{\phi(\vartheta x)} \\ &\overset{(*)}{=} \frac{2}{\vartheta}\lim_{x\to\infty}\frac{\phi(\vartheta x)\vartheta x + \Phi(\vartheta x)}{-\vartheta^2 x\phi(\vartheta x)} = -\frac{2}{\vartheta^3}\left(\vartheta + \lim_{x\to\infty}\frac{\Phi(\vartheta x)}{x\phi(\vartheta x)}\right) \\ &\overset{(*)}{=} -\frac{2}{\vartheta^3}\left(\vartheta + \lim_{x\to\infty}\frac{\phi(\vartheta x)\vartheta}{\phi(\vartheta x) - \vartheta^2 x^2\phi(\vartheta x)}\right) \\ &= -\frac{2}{\vartheta^3}\left(\vartheta + \lim_{x\to\infty}\frac{\vartheta}{1-\vartheta^2x^2}\right) = -\frac{2}{\vartheta^2},\end{aligned}$$

where the steps marked by (*) follow from the l'Hôpital's rule. Note that for $\vartheta < 0$, $\lim\limits_{x\to\infty}\Phi(\vartheta x)x = 0$ and $\lim\limits_{x\to\infty}x\phi(\vartheta x) = 0$.

References

[1] H. Finner, "A generalization of Hölder's inequality and some probability inequalities," *The Annals of Probability*, vol. 20, no. 4, pp. 1893–1901, 1992.