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Disjointness preserving relations on sets of uniformly continuous and Lipschitz functions ☆

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A B S T R A C T

For $i = 1, 2$, let X_i, E_i be topological spaces and let $A(X_i, E_i)$ be a set of continuous functions from X_i into E_i. A relation $R \subseteq A(X_1, E_1) \times A(X_2, E_2)$ is said to be a $\perp$-isomorphism if for any (f_1, f_2), (g_1, g_2), $(h_1, h_2) \in R$,

$$[f_1 = h_1] \cap [g_1 = h_1] = \emptyset \iff [f_2 = h_2] \cap [g_2 = h_2] = \emptyset.$$

The class of $\perp$-isomorphisms were introduced in [17] in the case where X_1, X_2 are locally compact Hausdorff spaces. They were shown to be applicable to problems such as the Banach-Stone Theorem, Kaplansky's Theorem on order preserving maps and the theorem of Gelfand-Kolmogorov on ring isomorphisms on spaces of continuous functions, plus many other results. The class of $\perp$-isomorphisms also include the well studied class of biseparating maps. In this paper, we consider $\perp$-isomorphisms in the case where X_i, E_i are complete metric spaces. Complete characterizations are given of $\perp$-isomorphisms between sets of uniformly continuous functions and sets of Lipschitz functions. The results apply whenever E_i is either a closed convex set in a Banach space or a compact connected Riemannian manifold. In particular, existence of a $\perp$-isomorphic relation between the spaces of uniformly continuous functions $U(X_1, E_1)$ and $U(X_2, E_2)$ implies that X_1 and X_2 are uniformly homeomorphic. An automatic continuity result is also obtained. However, in the case of Lipschitz functions, a $\perp$-isomorphism between $\mathrm{Lip}(X_1, E_1)$ and $\mathrm{Lip}(X_2, E_2)$ only induces a Lipschitz homeomorphism between certain suitably modified metrics, which reduce to the original metrics only in the case of bounded metrics.

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1. Introduction

The story began with three classical theorems. Let X be a compact Hausdorff space. The theorems of Banach-Stone [10,39], Kaplansky [31] and Gelfand-Kolmogorov [21] show that the space X is completely determined by either the isometric structure, or the order structure, or the ring structure of the space of continuous functions $C(X)$. Generalizations of these results abound. In the case of linear isometries, one can refer to Behrends [11] and the two volume monograph of Fleming and Jamison [19]. For order isomorphisms on spaces of real valued functions, refer to the authors' paper [35], which was preceded and motivated by [13–15], among others. With regard to rings of continuous functions, refer to [24] and the subsequent classic monograph [22]. These strands of investigations can be tied together by considering disjointness preserving operators. These are bijective mappings between spaces of functions so that two functions are disjoint if and only if their images are disjoint. For instance, Araujo [5] showed that a linear isometry between certain spaces of bounded vector-valued continuous functions or uniformly continuous functions are disjointness preserving; the recent paper [34] showed that the same holds for a linear isometry between certain spaces of vector-valued differentiable functions. It was shown in [36] that order preserving maps between many sublattices of spaces of continuous functions are also disjointness preserving. Copious research has been devoted to the study of disjointness preserving maps on various function spaces, such as spaces of continuous, uniformly continuous, Lipschitz and differentiable functions; see, e.g., [2–4], [6–9], [12], [20], [23–30], [33]. All of these papers are concerned with linear or additive maps between vector spaces of functions. Since additive bijective maps are $\mathbb{Q}$-linear, such maps are not far removed from the linear world. In [18], X. Feng studied nonlinear disjointness preserving maps, but stayed within the context of vector spaces of functions. As far as the authors know, the first investigation of disjointness preserving maps in a truly nonlinear setting is due to Cordeiro [17]. In the paper, the sets of functions involved need not carry any algebraic structure (e.g., as a vector space or a group). However, the paper [17] only deals with functions whose domains are locally compact Hausdorff spaces. In the present paper, we focus our attention on sets of functions defined on and with values in complete metric spaces, in particular, sets of uniformly continuous functions and sets of Lipschitz functions. Also, inspired by the recent paper [38], we consider disjointness preserving *relations* instead of mappings. Let X_i, E_i be complete metric spaces. Under the assumption that the range spaces E_i are arc length spaces that are locally uniformly contractible (see §4 for definitions), we completely characterized the $\perp$-isomorphisms (i.e., disjointness preserving relations; see §2) between the sets of uniformly continuous functions $U(X_1, E_1)$ and $U(X_2, E_2)$. A similar analysis is also carried out for sets of Lipschitz functions. Our results are valid in particular if E_i are complete convex sets in normed spaces or compact connected Riemannian manifolds.

Let us give an overview of the content of the paper. In §2, we recall the definition of $\perp$-isomorphism from [17,37], as well as the fundamental characterization of $\perp$-isomorphisms from [37] (Theorem 2.1). In §3, Theorem 3.2 gives a sufficient condition for when the Boolean isomorphism associated with a $\perp$-isomorphism induces a homeomorphism of the underlying domain spaces. Starting from §4, we specialize to sets of uniformly continuous functions or Lipschitz functions. The notions of uniformly as well as Lipschitz locally contractible metric spaces and are introduced. These turn out to provide the right contexts in which subsequent work can be carried out. Moreover, they are sufficiently general to include all (convex sets in) normed spaces as well as all compact connected Riemannian manifolds. In §4, many kinds of constructions of uniformly continuous and Lipschitz functions are given. These are all important for applications in the subsequent sections. Theorem 5.2 in §5 gives a representation for $\perp$-isomorphisms in the settings of uniformly continuous or Lipschitz functions. It paves the way for the characterizations of such $\perp$-isomorphisms given in Theorems 6.6 and Theorem 7.6. Moreover, Theorem 6.6 is applied to give a result on automatic continuity in the case of uniformly continuous functions (Theorem 6.7). In the final section, we prove the Lipschitzness (with respect to modified metrics) and the continuity (on a large subset) of the functions φ

and Φ associated with a $\perp$-isomorphism between Lipschitz spaces (Theorems 8.2 and 8.10). Both of these results require the use of the Baire Category Theorem at some point.

The cardinality of a set A is denoted by $|A|$.

2. $\perp$-isomorphisms

Let X, E be Hausdorff topological spaces. Denote the space of continuous functions from X into E by $C(X, E)$. Given $f, h \in C(X, E)$, define

$$[f \neq h] = \{x \in X : f(x) \neq h(x)\} \quad \text{and} \quad \sigma_h(f) = \operatorname{int} \overline{[f \neq h]}.$$

A subset $A(X, E)$ of $C(X, E)$ is *h-weakly regular* if

$$\Sigma_h := \{\sigma_h(f) : f \in A(X, E)\}$$

is a basis for the topology of X. If $f, g, h \in C(X)$, say that f and g are *disjoint with respect to h*, $f \perp_h g$, if $[f \neq h] \cap [g \neq h] = \emptyset$. Note that $f \perp_h g$ if and only if $\sigma_h(f) \cap \sigma_h(g) = \emptyset$.

Assume that X_i, E_i are Hausdorff topological spaces and $h_i \in C(X_i, E_i)$, $i = 1, 2$. A relation $R \subseteq C(X_1, E_1) \times C(X_2, E_2)$ is a *$\perp_{h_1,h_2}$-isomorphism* if for any $(f_1, f_2), (g_1, g_2) \in R$, $f_1 \perp_{h_1} g_1$ if and only if $f_2 \perp_{h_2} g_2$. Let π_i be the projection from $C(X_1, E_1) \times C(X_2, E_2)$ onto the i-th factor. R is *(h_1, h_2)-weakly regular* if $A(X_i, E_i) := \pi_i(R)$ is h_i-weakly regular in $C(X_i, E_i)$, $i = 1, 2$.

A set U in X_i is *regular open* if $U = \operatorname{int} \overline{U}$. Denote the set of all regular open sets in X_i by $\mathrm{RO}(X_i)$. Then $\mathrm{RO}(X_i)$ is a Boolean algebra with the operations $U_1 \wedge U_2 = U_1 \cap U_2$, $U_1 \vee U_2 = \operatorname{int} \overline{U_1 \cup U_2}$ and $\neg U = \operatorname{int}(U^c)$. Moreover, $\mathrm{RO}(X_i)$ is *complete* in the sense that every nonempty subset of $\mathrm{RO}(X_i)$ has a supremum.

Theorem 2.1. *[37, Theorem 2.4] Let R be (h_1, h_2)-weakly regular. Then R is a $\perp_{h_1,h_2}$-isomorphism if and only if there is a Boolean isomorphism $\theta : \mathrm{RO}(X_1) \to \mathrm{RO}(X_2)$ such that for any $(f_1, f_2) \in R$ and any $U \in \mathrm{RO}(X_1)$, $f_1 = h_1$ on U if and only if $f_2 = h_2$ on $\theta(U)$.*

Suppose that $(h_1, h_2) \in R \subseteq C(X_1, E_1) \times C(X_2, E_2)$. A homeomorphism $\varphi : X_1 \to X_2$ is *associated with* (R, h_1, h_2) if for any $(f_1, f_2) \in R$ and any $U \in \mathrm{RO}(X_1)$, $f = h_1$ on U if and only if $f_2 = h_2$ on $\varphi(U)$. In general, a $\perp_{h_1,h_2}$-isomorphism need not have a homeomorphism associated with it.

3. $\perp$-isomorphisms in the metric setting

For $i = 1, 2$, let X_i, E_i be complete metric spaces. In view of Theorem 2.1, one may ask if a Boolean isomorphism $\theta : \mathrm{RO}(X_1) \to \mathrm{RO}(X_2)$ always induces a homeomorphism $\varphi : X_1 \to X_2$. This turns out to be false. Indeed, in the proof of [17, Corollary 1.25], it was shown that $\mathrm{RO}([0, 1])$ and $\mathrm{RO}(S^1)$ are Boolean isomorphic, where S^1 denotes the unit circle in $\mathbb{R}^2$. Furthermore, J. Cabello Sánchez has shown that (in the case of complete metric spaces) $\mathrm{RO}(X_1)$ and $\mathrm{RO}(X_2)$ are Boolean isomorphic if and only if there is a complete metric space Z so that Z is homeomorphic to dense subspaces of X_1 and X_2 respectively; see [16, Theorem 2.14]. Therefore, additional properties have to be imposed on a $\perp$-isomorphism in order to guarantee that the associated Boolean isomorphism gives rise to a homeomorphism. This is the main result of the section (Theorem 3.2). Assume that $R \subseteq C(X_1, E_1) \times C(X_1, E_2)$ and set $A(X_i, E_i) = \pi_i(R)$. Set

$$\mathcal{A}_R = \{(h_1, h_2) \in R : R \text{ is a } (h_1, h_2)\text{-weakly regular } \perp_{h_1,h_2}\text{-isomorphism}\}.$$

Definition 3.1. Assume that $(h_1, h_2), (k_1, k_2) \in \mathcal{A}_R$. Say that $(k_1, k_2) \ll (h_1, h_2)$ if

(1) For any $U \in \mathrm{RO}(X_1)$, $x \notin \overline{U}$, there exist $(f_1, f_2) \in R$ and $U' \in \mathrm{RO}(X_1)$ so that $x \in U'$ and that $f_1 = h_1$ on U and $f_1 = k_1$ on U'.
(2) For any $V \in \mathrm{RO}(X_2)$, $y \notin \overline{V}$, there exist $(g_1, g_2) \in R$ and $V' \in \mathrm{RO}(X_2)$ so that $y \in V'$ and that $g_2 = h_2$ on V and $g_2 = k_2$ on V'.
(3) Given a separated sequence (x_n) in X_1, there are infinite subsets I, J of $\mathbb{N}$, a sequence $(U_n)_{n \in I \cup J}$ in $\mathrm{RO}(X_1)$ and $(f_1, f_2) \in R$ so that $x_n \in U_n$ for all $n \in I \cup J$, $f_1 = h_1$ on U_n for all $n \in I$ and $f_1 = k_1$ on U_n for all $n \in J$.
(4) Given a separated sequence (y_n) in X_2, there are infinite subsets I, J of $\mathbb{N}$, a sequence $(V_n)_{n \in I \cup J}$ in $\mathrm{RO}(X_2)$ and $(g_1, g_2) \in R$ so that $y_n \in V_n$ for all $n \in I \cup J$, $g_2 = h_2$ on V_n for all $n \in I$ and $g_2 = k_2$ on V_n for all $n \in J$.

Theorem 3.2. *Assume that $(h_1, h_2) \in R$ and that R is a (h_1, h_2)-weakly regular $\perp_{h_1,h_2}$-isomorphism. Suppose that for any $x \in X_1$ and $y \in X_2$, there exist $(k_1, k_2), (l_1, l_2) \ll (h_1, h_2)$ and that $k_1(x) \neq h_1(x)$, $l_2(y) \neq h_2(y)$. Then there exists a homeomorphism $\varphi : X_1 \to X_2$ associated with (R, h_1, h_2).*

Proof. By Theorem 2.1, there is a Boolean isomorphism $\theta : \mathrm{RO}(X_1) \to \mathrm{RO}(X_2)$ such that for any $(f_1, f_2) \in R$ and any $U \in \mathrm{RO}(X_1)$, $f_1 = h_1$ on U if and only if $f_2 = h_2$ on $\theta(U)$. Let $x_0 \in X_1$. By assumption, there exists $(k_1, k_2) \ll (h_1, h_2)$ so that $k_1(x_0) \neq h_1(x_0)$.

Claim. $\bigcap\{\overline{\theta(U)} : x_0 \in U \in \mathrm{RO}(X_1)\}$ contains exactly 1 point.

Let (U_n) be a sequence in $\mathrm{RO}(X_1)$ so that $x_0 \in U_n$ for each n and that $\operatorname{diam} U_n \to 0$. Pick $y_n \in \theta(U_n)$ for each n. Suppose that (y_n) has a separated subsequence. By Definition 3.1(4), there are infinite subsets I, J of $\mathbb{N}$, a sequence $(V_n)_{n \in I \cup J}$ in $\mathrm{RO}(X_2)$ and $(g_1, g_2) \in R$ so that $y_n \in V_n$ for all $n \in I \cup J$, $g_2 = h_2$ on V_n for all $n \in I$ and $g_2 = k_2$ on V_n for all $n \in J$. Thus $g_1 = h_1$ on $\theta^{-1}(V_n)$, $n \in I$, and by [37, Proposition 2.7], $g_1 = k_1$ on $\theta^{-1}(V_n)$, $n \in J$. If $n \in I \cup J$, $y_n \in \theta(U_n) \cap V_n$ and thus $U_n \cap \theta^{-1}(V_n) \neq \emptyset$. Let x_n be a point in the intersection. As I, J are infinite sets, $x_0 \in U_n$ and $\operatorname{diam} U_n \to 0$, $(x_n)_{n \in I}$ and $(x_n)_{n \in J}$ both converge to x_0. However, $g_1(x_n) = h_1(x_n)$, $n \in I$, and $g_1(x_n) = k_1(x_n)$, $n \in J$. In the limit, $h_1(x_0) = g_1(x_0) = k_1(x_0)$, contrary to the choice of k_1. We thus conclude that (y_n) has an accumulation point. Denote it by y_0. If $x_0 \in U \in \mathrm{RO}(X_1)$, then $U_n \subseteq U$ for all but finitely many n. It follows that $y_0 \in \overline{\theta(U)}$. This proves that $\bigcap\{\overline{\theta(U)} : x_0 \in U \in \mathrm{RO}(X_1)\}$ is nonempty.

Suppose, if possible, that there are distinct points $y_1, y_2 \in \bigcap\{\overline{\theta(U)} : x_0 \in U \in \mathrm{RO}(X_1)\}$. Choose $V_1 \in \mathrm{RO}(X_2)$ so that $y_1 \in V_1$ and that $y_2 \notin \overline{V_1}$. With (k_1, k_2) as above, we have $(k_1, k_2) \ll (h_1, h_2)$. Hence there exists $(g_1, g_2) \in R$ and an open set $V_2 \in \mathrm{RO}(X_2)$ so that $y_2 \in V_2$, $g_2 = h_2$ on V_1 and that $g_2 = k_2$ on V_2. If $x_0 \in U \subseteq \mathrm{RO}(X_1)$, then $V_i \cap \theta(U) \neq \emptyset$ and thus $\theta^{-1}(V_i) \cap U \neq \emptyset$, $i = 1, 2$. Hence $x_0 \in \bigcap_{i=1}^{2} \overline{\theta(V_i)}$. By choice of θ, $g_1 = h_1$ on $\theta^{-1}(V_1)$. Also, by [37, Proposition 2.7], $g_1 = k_1$ on $\theta^{-1}(V_2)$. Therefore, $h_1(x_0) = g_1(x_0) = k_1(x_0)$, contrary to the choice of k_1. This completes the proof of the claim.

Define $\varphi : X_1 \to X_2$ and $\psi : X_2 \to X_1$ so that

$$\{\varphi(x)\} = \bigcap\{\overline{\theta(U)} : x \in U \in \mathrm{RO}(X_1)\},$$

$$\{\psi(y)\} = \bigcap\{\overline{\theta^{-1}(V)} : y \in V \in \mathrm{RO}(X_2)\}.$$

φ and ψ are well defined by the claim and symmetry. If $\varphi(x) = y$ and $y \in V \in \mathrm{RO}(X_2)$, then $V \cap \theta(U) \neq \emptyset$ if $x \in U \in \mathrm{RO}(X_1)$. Thus $\theta^{-1}(V) \cap U \neq \emptyset$. Hence $x \in \overline{\theta^{-1}(V)}$. This proves that $\psi(y) = x$. By symmetry, φ and ψ are mutual inverses.

Next, we show the continuity of φ. Continuity of ψ follows by symmetry. Suppose that (x_n) is a sequence in X_1 converging to x. We can choose a sequence (U_n) in $\mathrm{RO}(X_1)$ so that $x_0, x_n \in U_n$ for each n and that $\operatorname{diam} U_n \to 0$. By definition, $\varphi(x_n) \in \overline{\theta(U_n)}$. Pick $y_n \in \theta(U_n)$ so that $d(y_n, \varphi(x_n)) \to 0$. By the first paragraph of the proof of the Claim, $\varphi(x)$ is an accumulation point of the sequence (y_n) and thus an accumulation point of the sequence $(\varphi(x_n))$. Apply the argument to any subsequence of (x_n) to see that every subsequence (x'_n) of (x_n) has a further subsequence (x''_n) so that $(\varphi(x''_n))$ converges to $\varphi(x)$. Thus $(\varphi(x_n))$ converges to $\varphi(x)$, as required.

Finally, if $U \in \mathrm{RO}(X_1)$, it follows from the definition of φ that $\varphi(U) \subseteq \overline{\theta(U)}$. Since $\varphi(U)$ is open and $\theta(U)$ is regular open, $\varphi(U) \subseteq \theta(U)$. By symmetry, $\varphi^{-1}(\theta(U)) \subseteq U$. Hence $\theta(U) \subseteq \varphi(U)$. It follows that φ is associated with (R, h_1, h_2). $\square$

4. Construction of uniformly continuous or Lipschitz functions

Let X, E be metric spaces. For simplicity, the metric on any given metric space will be denoted by the same symbol d unless stated otherwise. A *(continuous) path* in E is a continuous function $\gamma : [a, b] \to E$, where $[a, b]$ is a closed bounded interval in $\mathbb{R}$. In this case, we say that γ goes from $\gamma(a)$ to $\gamma(b)$. The *length* of γ is

$$l(\gamma) := \sup\{\sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)) : n \in \mathbb{N}, a = t_0 < \cdots < t_n = b\} \in [0, \infty].$$

E is *finitely path connected* if for any $u, v \in E$, there exists a path γ that goes from u to v with $l(\gamma) < \infty$. In this case, we define the *arc length metric* d_l on E by

$$d_l(u, v) = \inf\{l(\gamma) : \gamma \text{ is a path in } E \text{ that goes from } u \text{ to } v\}.$$

E is an *arc length space* if it is finitely path connected and $d(u, v) = d_l(u, v)$ for any $u, v \in E$. Suppose that u, v are points in an arc length space E. There is a path γ in E from u to v so that $l(\gamma) \le 2d(u, v)$. Reparametrize γ by arc length to assume without loss of generality that $\gamma : [0, l(\gamma)] \to E$ is a 1-Lipschitz path that goes from u to v. Define $\overline{\gamma} : [0, d(u, v)] \to E$ by $\overline{\gamma}(t) = \gamma(\frac{l(\gamma)t}{d(u,v)})$. Then $\overline{\gamma} : [0, d(u, v)] \to E$ is a 2-Lipschitz path that goes from u to v.

For any $f : X \to E$, define its modulus of continuity to be $\omega_f : [0, \infty) \to [0, \infty]$, where

$$\omega_f(t) = \sup\{d(f(x_1), f(x_2)) : x_1, x_2 \in X, d(x_1, x_2) \le t\}.$$

Obviously, ω_f is a nondecreasing function. f is uniformly continuous ($f \in U(X, E)$) if and only if $\lim_{t\downarrow 0} \omega_f(t) = 0$; it is Lipschitz ($f \in \mathrm{Lip}(X, E)$) if and only if there exists $C < \infty$ so that $\omega_f(t) \le Ct$. For $f \in \mathrm{Lip}(X, E)$, the smallest C such that $\omega_f(t) \le Ct$ is the Lipschitz constant of f, denoted by $\mathrm{Lip}(f)$.

Let Ω be the set of nondecreasing functions $\omega : [0, \infty) \to [0, \infty]$ such that $\lim_{t\downarrow 0} \omega(t) = 0$. E is *uniformly locally contractible (in E)* (with parameters ω and M) if there exist $\omega \in \Omega$ and $M > 0$ with $\omega(M) < \infty$ so that for any $0 < r \le M$ and any $u \in E$, there is a function $H : [0, \omega(r)] \times B(u, r) \to E$ so that

$$\omega_H \le \omega, H(0, v) = v \text{ and } H(\omega(r), v) = u \text{ for all } v \in B(u, r).$$

(To be specific, we take the metric on $[0, \omega(r)] \times B(u, r)$ to be the sum of the metrics on the components.) If there is a constant $C < \infty$ so that $\omega(t) \le Ct$, then E is said to be *Lipschitz locally contractible (in E)*.

Example 4.1.

(1) If E is a convex set in a normed space, then it is an arc length space. The space E is Lipschitz locally contractible.
(2) If E is a compact connected Riemannian manifold (with the arc length metric), then it is Lipschitz locally contractible. For definitions and results on Riemannian manifolds, refer to [32].

Proof. (1) is clear.

In order to prove (2), we first make a claim. Denote by B the set $\{x \in \mathbb{R}^k : \|x\|_2 < 1\}$.

Claim. Let E be a Riemannian manifold. Then every point p in E has an open neighborhood that is Lipschitz homeomorphic to $B \subseteq \mathbb{R}^k$, where k is the dimension of E.

Let $p \in E$ and let $\xi = (x^1, \dots, x^k) : V \to \xi(V) \subseteq \mathbb{R}^k$ be an admissible chart so that $p \in V$ and $\xi(p) = 0$. The Riemannian metric g can be written as $g = g_{ij} dx^i \otimes dx^j$ (summation convention) on the set V, where g_{ij} are smooth function on V and $A_q := (g_{ij}(q))_{i,j}$ is a positive definite matrix for any $q \in V$. By shrinking V if necessary, we may assume that there exists $1 \le c < \infty$ so that $\frac{\|u\|_2^2}{c} \le \langle A_q u, u\rangle \le c\|u\|_2^2$ for any $q \in V$ and any $u \in \mathbb{R}^k$. Here $\langle \cdot, \cdot \rangle$ is the usual inner product on $\mathbb{R}^k$. Suppose that $\gamma : [a, b] \to V$ is a smooth path. Then $\gamma'(t) = (x^i \circ \gamma)'(t)\partial_{x^i}|_{\gamma(t)}$. Thus, for any $t \in [a, b]$,

$$\frac{\|u(t)\|_2^2}{c} \le g_{\gamma(t)}(\gamma'(t), \gamma'(t)) = \langle A_{\gamma(t)} u(t), u(t)\rangle \le c\|u(t)\|_2^2,$$

where $u(t) = ((x^i \circ \gamma)'(t))_i \in \mathbb{R}^k$. Then

$$\frac{l(\xi \circ \gamma)}{c} \le l(\gamma) = \int_a^b \sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))}\, dt \le c l(\xi \circ \gamma).$$

Choose $r > 0$ so that $B(p, 5r) \subseteq V$, where the ball $B(p, 5r)$ is taken with respect to the arc length metric. If $q_1, q_2 \in B(p, r)$, there is a piecewise smooth path γ of length at most $2d(q_1, q_2) < 4r$ that goes from q_1 to q_2. If q is a point on γ, then

$$d(q, p) \le d(q, q_1) + d(q_1, p) < l(\gamma) + r < 5r.$$

Thus the path γ lies in $B(p, 5r) \subseteq V$. By the above, $\xi \circ \gamma$ is a piecewise smooth path in $\mathbb{R}^k$ from $\xi(q_1)$ to $\xi(q_2)$ so that $l(\xi \circ \gamma) \le c l(\gamma) \le 2cd(q_1, q_2)$. Thus $\|\xi(q_1) - \xi(q_2)\|_2 \le 2cd(q_1, q_2)$. So $\xi : B(p, r) \to \xi(B(p, r)) \subseteq \mathbb{R}^k$ is Lipschitz. Since $\xi(p) = 0$, there exists $s > 0$ so that $W := \{x \in \mathbb{R}^k : \|x\|_2 < s\} \subseteq \xi(B(p, r))$. If $x_1, x_2 \in W$, let Γ be the straight line path in $\mathbb{R}^k$ connecting x_1 to x_2, with $l(\Gamma) = \|x_1 - x_2\|_2$. Then $\gamma := \xi^{-1} \circ \Gamma$ is a piecewise smooth path in $B(p, r)$ connecting $\xi^{-1}(x_1)$ to $\xi^{-1}(x_2)$. By the above, $l(\gamma) \le c l(\xi \circ \gamma) = c l(\Gamma)$. Hence $d(\xi^{-1}(x_1), \xi^{-1}(x_2)) \le l(\gamma) \le c\|x_1 - x_2\|_2$. This proves that ξ^{-1} is Lipschitz on W. Since ξ is Lipschitz on $\xi^{-1}(W) \subseteq B(p, r)$, $\xi : U := \xi^{-1}(W) \to W$ is a Lipschitz homeomorphism. Since all open balls in $\mathbb{R}^k$ are Lipshcitz homeomorphic, the claim follows.

Now let E be a compact connected Riemannian manifold. By the claim, for each $p \in E$, there is an open neighborhood U_p of p and a Lipschitz homeomorphism $\xi_p : U_p \to B$. For each p, choose $R_p > 0$ so that $B(p, 2R_p) \subseteq U_p$. There are $p_1, \dots, p_m$ so that $E = \bigcup_{i=1}^m B(p_i, R_{p_i})$ By the Lebesgue Covering Lemma, there exists $\delta > 0$ so that for any $u \in E$, $B(u, \delta)$ is contained in some $B(p_i, R_{p_i})$, $1 \le i \le m$. Choose M so that $0 < M < \min\{\delta, R_{p_1}, \dots, R_{p_m}\}$. Let $L = \max\{\mathrm{Lip}(\xi_{p_i}), \mathrm{Lip}(\xi_{p_i}^{-1}) : 1 \le i \le m\}$ and let $\omega(t) = L^2 t$. Assume that $u \in E$ and $0 < r \le M$. Choose i such that $B(u, \delta) \subseteq B(p_i, R_{p_i})$. If $v \in B(u, r)$, then $d(v, p_i) < r + d(u, p_i) < 2R_{p_i}$. Hence $B(u, r) \subseteq U_{p_i}$ and thus $\xi_{p_i}(B(u, r)) \subseteq B$. Therefore, the function $H : [0, L^2 r] \times B(u, r) \to E$ given by

$$H(t, v) = \xi_{p_i}^{-1}[(1 - \frac{t}{L^2 r})\xi_{p_i}(v) + \frac{t}{L^2 r}\xi_{p_i}(u)]$$

is well defined. Clearly, $H(L^2 r, v) = u$ and $H(0, v) = v$ for any $v \in B(u, r)$. Suppose that $(t_1, v_1), (t_2, v_2) \in [0, L^2 r] \times B(u, r)$. Then

$$\begin{aligned}
&\|\xi_{p_i}(H(t_1,v_1)) - \xi_{p_i}(H(t_2,v_2))\|_2 \\
&\qquad = \|(1-\frac{t_1}{L^2 r})(\xi_{p_i}(v_1) - \xi_{p_1}(v_2)) + \frac{t_1-t_2}{L^2 r}(\xi_{p_i}(v_2) - \xi_{p_i}(u))\|_2 \\
&\qquad \le \|\xi_{p_i}(v_1) - \xi_{p_i}(v_2)\|_2 + |t_1-t_2|\,\frac{Ld(v_2,u)}{L^2 r} \\
&\qquad \le Ld(v_1,v_2) + \frac{|t_1-t_2|}{L} \\
&\qquad \le L(d(v_1,v_2) + |t_1-t_2|),\ \text{since } L \ge 1.
\end{aligned}$$

Hence

$$d(H(t_1,v_1), H(t_2,v_2)) \le L^2(d(v_1,v_2) + |t_1-t_2|).$$

That is, $\omega_H(t) \le L^2 t$. This completes the proof of (2). □

The local contractibility properties introduced above are crucial for our purposes because they allow for various constructions of uniformly continuous or Lipschitz functions. The basic prototype of such a construction is given in the next lemma.

Lemma 4.2. *Assume that E is an arc length space that is uniformly locally contractible with parameters ω and M. Let $x_0 \in X$, $s > 0$, $0 < r \le M$ and let $h_i : B(x_0, 7s) \to B(h_i(x_0), r) \subseteq E$, $i = 1, 2$. If $x \in B(x_0, 7s)$, set $|x| = d(x, x_0)$. Also let $a = d(h_1(x_0), h_2(x_0))$ and $b = \omega(r)$. There is a function $f : B(x_0, 7s) \to E$ such that*

(a)

$$f(x) = \begin{cases} h_1(x) & \text{if } |x| \in [0,s), \\ h_2(x) & \text{if } |x| \in [6s,7s). \end{cases} \tag{4.1}$$

(b) *If $x \in B(x_0, 7s)$, then*

$$d(f(x), h_2(x)) \le 2(\omega(b) + a) + \omega_{h_2}(5s).$$

(c) *If $x_1, x_2 \in B(x_0, 7s)$ and $c = d(x_1, x_2)$, then*

$$\begin{aligned}
d(f(x_1), f(x_2)) \le\, & 4\omega[\frac{bc}{s} \wedge b + \omega_{h_1}(c) + \omega_{h_2}(c)] + 4(\frac{ac}{s} \wedge a) \\
& + 3\omega_{h_2}(5c).
\end{aligned}$$

Proof. There is a 2-Lipschitz path $\gamma : [0, a] \to E$ that goes from $h_1(x_0)$ to $h_2(x_0)$. There are functions $H_i : [0, b] \times B(h_i(x_0), r) \to E$ so that $\omega_{H_i} \le \omega$,

$$H_i(0, v) = v \text{ and } H_i(b, v) = h_i(x_0) \text{ for all } v \in B(h_i(x_0), r).$$

Let $\alpha : [0, 3s] \to [0, b]$ be the function $\alpha(t) = b[(\frac{t}{s} - 1) \wedge 1]^+$. Thus α is a continuous function that is equal to 0 and b respectively on $[0, s]$ and $[2s, 3s]$ respectively, and affine on $[s, 2s]$ with slope $\frac{b}{s}$. Define $f : B(x_0, 7s) \to E$ by

$$f(x)=\begin{cases}H_1(\alpha(|x|),h_1(x)) & \text{if } |x|<3s,\\ \gamma(\frac{a}{b}\alpha(|x|-2s)) & \text{if } 2s<|x|<5s,\\ H_2(b-\alpha(|x|-4s),h_2(x)) & \text{if } 4s<|x|<7s.\end{cases}$$

It is easy to check that f is well defined and that f satisfies (4.1).

(b) $|x|\in[0,3s)$, then

$$h_i(x)=H_i(\alpha(s),h_i(x)) \text{ and } h_i(x_0)=H_i(\alpha(2s),h_i(x)), i=1,2.$$

Hence

$$\begin{aligned}d(f(x),h_2(x)) &\le d(H_1(\alpha(|x|),h_1(x)),H_1(\alpha(2s),h_1(x)))+d(h_1(x_0),h_2(x_0))+\\ &\quad +d(H_2(\alpha(2s),h_2(x)),H_2(\alpha(s),h_2(x)))\\ &\le \omega_{H_1}(b)+a+\omega_{H_2}(b)\\ &\le 2\omega(b)+a.\end{aligned}$$

If $|x|\in(4s,7s)$, then

$$\begin{aligned}d(f(x),h_2(x)) &\le d(H_2(b-\alpha(|x|-4s)),h_2(x)),H_2(b-\alpha(2s),h_2(x)))\\ &\le \omega_{H_2}(b)\le\omega(b).\end{aligned}$$

Finally, if $|x|\in(2s,5s)$, then

$$\begin{aligned}d(f(x),h_2(x)) &= d(\gamma(\frac{a}{b}\alpha(|x|-2s)),\gamma(a))+d(h_2(x_0),h_2(x))\\ &\le l(\gamma)+\omega_{h_2}(|x|)\le 2a+\omega_{h_2}(5s).\end{aligned}$$

(c) We will repeatedly use the fact that

$$|\alpha(t_1)-\alpha(t_2)|\le\frac{b}{s}|t_1-t_2|\wedge b \text{ if } t_1,t_2\in[0,3s].$$

Suppose that $|x_1|,|x_2|\in[0,3s)$. Then $|\alpha(|x_1|)-\alpha(|x_2|)|\le\frac{bc}{s}\wedge b$. Hence

$$d(f(x_1),f(x_2))\le\omega_{H_1}(\frac{bc}{s}\wedge b+d(h_1(x_1),h_1(x_2)))\le\omega(\frac{bc}{s}\wedge b+\omega_{h_1}(c)).$$

If $|x_1|,|x_2|\in(2s,5s)$, then $|\alpha(|x_1|-2s)-\alpha(|x_2|-2s)|\le\frac{bc}{s}\wedge b$. Hence, keeping in mind that γ is 2-Lipschitz,

$$d(f(x_1),f(x_2))\le 2(\frac{ac}{s}\wedge a).$$

If $|x_1|,|x_2|\in(4s,7s)$, then $|\alpha(|x_1|-4s)-\alpha(|x_2|-4s)|\le\frac{bc}{s}\wedge b$. Hence

$$d(f(x_1),f(x_2))\le\omega_{H_2}(\frac{bc}{s}\wedge b+d(h_2(x_1),h_2(x_2)))\le\omega(\frac{bc}{s}\wedge b+\omega_{h_2}(c)).$$

This proves (c) if $|x_1|,|x_2|$ both belong to one of the intervals $[0,3s),[2s,5s)$ or $(4s,7s)$. Otherwise, we must have $s\le d(x_1,x_2)=c$. Using (b), we have

$$\begin{aligned} d(f(x_1), f(x_2)) &\le d(f(x_1), h_2(x_1)) + d(h_2(x_1), h_2(x_2)) + d(h_2(x_2), f(x_2)) \\ &\le 4(\omega(b) + a) + 2\omega_{h_2}(5s) + d(h_2(x_1), h_2(x_2)) \\ &\le 4\omega(b) + 4a + 3\omega_{h_2}(5c). \end{aligned}$$

In the present case, $b = \frac{bc}{s} \wedge b$ and $a = \frac{ac}{s} \wedge a$. Thus (c) holds. □

Proposition 4.3. *Assume that E is an arc length space that is uniformly locally contractible with parameters ω and M. Let $(x_n)_{n\in I}$ be a finite or infinite sequence of distinct points in X. Suppose that*

(a) *(s_n) is a positive sequence so that $r_n := \omega(8s_n) \le M$ and that the sets $B(x_n, 8s_n)$ are pairwise disjoint;*
(b) *$h_n : B(x_n, 7s_n) \to E$, $n \in I$, and $k : X \to E$ are functions so that $\omega_{h_n}, \omega_k \le \omega$.*

Set $a_n = d(h_n(x_n), k(x_n))$ and $b_n = \omega(r_n)$. Then there is a function $f : X \to E$ so that
(1)

$$f(x) = \begin{cases} h_n(x) & \text{if } 0 \le d(x, x_n) < s_n \text{ for some } n, \\ k(x) & \text{if } x \notin \bigcup_{n\in I} B(x_n, 6s_n). \end{cases}$$

(2) *For any $z_1, z_2 \in X$, let $c = d(z_1, z_2)$. Then*

$$d(f(z_1), f(z_2)) \le 4 \sup_n \{\omega[\frac{b_n c}{s_n} \wedge b_n + 2\omega(c)] + \big(\frac{a_n c}{s_n} \wedge a_n\big)\} + 3\omega(5c).$$

Proof. For each n, apply Lemma 4.2 with the pair of functions h_n and k on the set $B(x_n, 7s_n)$. We obtain a function $f_n : B(x_n, 7s_n) \to E$ so that

$$f_n(x) = \begin{cases} h_n(x) & \text{if } d(x, x_n) < s_n, \\ k(x) & \text{if } 6s_n \le d(x, x_n) < 7s_n \end{cases}$$

and that for $z_1, z_2 \in B(x_n, 7s_n)$,

$$d(f_n(z_1), k(z_1))) \le 2(\omega(b_n) + a_n) + \omega(5s_n), \tag{4.2}$$

$$\begin{aligned} d(f_n(z_1), f_n(z_2)) \le 4\omega[\frac{b_n c}{s_n} \wedge b_n + 2\omega(c)] + 4(\frac{a_n c}{s_n} \wedge a_n) \\ + 3\omega(5c), \end{aligned} \tag{4.3}$$

where $c = d(z_1, z_2)$. Define $f : X \to E$ by

$$f(x) = \begin{cases} f_n(x) & \text{if } x \in B(x_n, 7s_n) \text{ for some } n, \\ k(x) & \text{if } x \notin \bigcup B(x_n, 7s_n). \end{cases}$$

Condition (1) of the proposition holds by definition of f.

Let us check condition (2). Let $z_1, z_2 \in X$ and $c = d(z_1, z_2)$.

Case a. $z_1, z_2 \notin \bigcup_{n\in I} B(x_n, 6s_n)$. We have

$$d(f(z_1), f(z_2)) = d(k(z_1), k(z_2)) \le \omega(c).$$

Thus (2) holds.

For the remaining cases, we may assume without loss of generality that $z_1 \in B(x_m, 6s_m)$ for some m.

Case b. $z_1 \in B(x_m, 6s_m)$ for some m and $z_2 \notin \bigcup_{n\in I} B(x_n, 7s_n)$. Then $c \geq s_m$ and thus $\frac{b_m c}{s_m} \wedge b_m = b_m$ and $\frac{a_m c}{s_m} \wedge a_m = a_m$. Now

$$\begin{aligned} d(f(z_1), f(z_2)) &= d(f_m(z_1), k(z_2)) \leq d(f_m(z_1), k(z_1)) + d(k(z_1), k(z_2)) \\ &\leq 2(\omega(b_m) + a_m) + \omega(5s_m) + \omega(c) \quad \text{by (4.2)} \\ &\leq 2\omega(b_m) + 2a_m + 3\omega(5c). \end{aligned}$$

Hence (2) follows.

Case c. $z_1 \in B(x_m, 6s_m)$ and $z_2 \in B(x_l, 7s_l)$, $m \neq l$. Employing (4.2) with $n = m$ and $n = l$ respectively,

$$\begin{aligned} d(f(z_1), f(z_2)) &\leq d(f_m(z_1), k(z_1)) + d(k(z_1), k(z_2)) + d(k(z_2), f_l(z_2)) \\ &\leq [2(\omega(b_m) + a_m) + \omega(5s_m)] + [2(\omega(b_l) + a_l) + \omega(5s_l)] + \omega(c). \end{aligned}$$

Since $B(x_m 8s_m) \cap B(x_l, 8s_l) = \emptyset$, $c \geq s_l \vee s_m$. Thus

$$\frac{b_m c}{s_m} \wedge b_m = b_m, \frac{a_m c}{s_m} \wedge a_m = a_m, \frac{b_l c}{s_l} \wedge b_l = b_l, \frac{a_l c}{s_l} \wedge a_l = a_l.$$

Hence (2) holds in this case.

Case d. $z_1 \in B(x_m, 6s_m)$ and $z_2 \in B(x_m, 7s_m)$. Then $f(z_i) = f_m(z_i)$, $i = 1, 2$. So (2) follows from (4.3). □

Corollary 4.4. *Let the notation and assumptions be as in Proposition 4.3.*

(1) *If (s_n), (b_n) and (a_n) are all null sequences, then $f \in U(X, E)$.*
(2) *If $(\frac{b_n}{s_n})$ and $(\frac{a_n}{s_n})$ are bounded sequences, then $f \in U(X, E)$.*
(3) *Assume that there exists $C < \infty$ so that $\omega(t) \leq Ct$ and that the sequence $(\frac{a_n}{s_n})$ is bounded. Then $f \in \mathrm{Lip}(X, E)$.*

Proof. (1) For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ so that $b_n, a_n < \varepsilon$ if $n > N$. Then

$$4 \sup_{n>N} \{\omega[\frac{b_n c}{s_n} \wedge b_n + 2\omega(c)] + (\frac{a_n c}{s_n} \wedge a_n)\} \leq 4\omega(\varepsilon + 2\omega(c)) + 4\varepsilon.$$

By (2) of Proposition 4.3, $\limsup_{c\downarrow 0} \omega_f(c) \leq 4\omega(\varepsilon) + 4\varepsilon$. Thus $\lim_{c\downarrow 0} \omega_f(c) = 0$ and so $f \in U(X, E)$.

(2) Let A be such that $b_n, a_n \leq As_n$. By Proposition 4.3,

$$\omega_f(c) \leq 4[\omega(Ac + 2\omega(c)) + Ac] + 3\omega(5c).$$

Thus $f \in U(X, E)$.

(3) Note that $b_n = \omega(\omega(8s_n)) \leq 8C^2 s_n$. As in (2), choose A so that $a_n \leq As_n$. Then by Proposition 4.3

$$\omega_f(c) \leq 4[\omega(8C^2 c + 2\omega(c)) + Ac] + 3\omega(5c).$$

So $f \in \mathrm{Lip}(X, E)$. □

Corollary 4.5. *Assume that E is an arc length space that is uniformly locally contractible. Let $k \in U(X,E)$ and let $(x_n)_{n\in I}$ be a finite or infinite separated sequence in X. Suppose either*

(1) *$h \in U(X,E)$ and that $\sup_{n\in I} d(h(x_n), k(x_n)) < \infty$; or*
(2) *$(u_n)_{n\in I} \subseteq E$ satisfies $\sup_{n\in I} d(u_n, k(x_n)) < \infty$.*

For any $\eta > 0$, there exist $0 < s < \eta$ and $f \in U(X,E)$ so that

$$f = \begin{cases} h, \text{ respectively } u_n, & \text{on } B(x_n, s) \text{ for each } n, \\ k & \text{outside } \bigcup_{n\in I} B(x_n, \eta). \end{cases}$$

Moreover, if E is Lipschitz locally contractible and h, k are Lipschitz functions, then f can be chosen from $\mathrm{Lip}(X,E)$.

Proof. There are ω and M so that E is uniformly local contractible with parameters ω and M and that $\omega_h, \omega_k \le \omega$. Choose $0 < s < \eta/6$ so that $\omega(8s) \le M$ and that the sets $B(x_n, 8s)$, $n \in I$, are pairwise disjoint. Apply Proposition 4.3 with $s_n = s$ and $h_n = h|_{B(x_n,7s)}$, respectively, $h_n = u_n$ on $B(x_n, 7s)$, $n \in I$. It follows from (1) of Proposition 4.3 that $f = h$, respectively u_n, on $B(x_n, s)$ for each n and that $f = k$ outside $\bigcup_{n\in I} B(x_n, \eta)$. By definition, $b_n = \omega(\omega(8s))$ for all n and hence $(\frac{b_n}{s_n}) = (\frac{\omega(\omega(8s))}{s})$ is bounded. Also, $a_n = d(h(x_n), k(x_n))$, respectively, $d(u_n, k(x_n))$. Thus $(\frac{a_n}{s_n}) = (\frac{a_n}{s})$ is bounded by assumption. The desired conclusions follow from Corollary 4.4(2) and (3) respectively. $\square$

Corollary 4.6. *Assume that E is an arc length space that is uniformly locally contractible. Let $k \in U(X,E)$ and let $(x_n)_{n\in\mathbb{N}}$ be a separated sequence in X. Suppose that $(x'_n)_{n\in\mathbb{N}} \subseteq X$ satisfies $0 < d(x_n, x'_n) \to 0$ and that (u_n) is a sequence in E. If $d(u_n, k(x_n)) \to 0$, then there exists $f \in U(X,E)$ and an infinite set $I \subseteq \mathbb{N}$ so that $f(x_n) = u_n$ and $f(x'_n) = k(x'_n)$ for all $n \in I$. Furthermore, if E is Lipschitz locally contractible, $k \in \mathrm{Lip}(X,E)$ and $(\frac{d(u_n,k(x_n))}{d(x_n,x'_n)})$ is bounded, then f can be chosen from* $\mathrm{Lip}(X,E)$.

Proof. Let ω and M be the parameters for uniform local contractibility of E. We may assume that $\omega_k \le \omega$. Let $s_n = \frac{d(x_n,x'_n)}{7}$. By removing a finite number of terms, we may assume that the sets $B(x_n, 8s_n)$, $n \in \mathbb{N}$, are pairwise disjoint and that $r_n := \omega(8s_n) \le M$. Apply Proposition 4.3 with $h_n = u_n$ on $B(x_n, 7s_n)$ to obtain $f : X \to E$ such that $f(x_n) = h_n(x_n) = u_n$ and that $f(x'_n) = k(x'_n)$. In the notation of the proposition, $b_n = \omega(\omega(8s_n)) \to 0$. By assumption, $a_n \to 0$. Hence $f \in U(X,E)$ by Corollary 4.4(1).

Suppose that E is Lipschitz locally contractible, $k \in \mathrm{Lip}(X,E)$ and $(\frac{d(u_n,k(x_n))}{d(x_n,x'_n)})$ is bounded. Then there exists $C < \infty$ so that $\omega(t) \le Ct$. Furthermore $(\frac{a_n}{s_n})$ is bounded. Hence $f \in \mathrm{Lip}(X,E)$ by Corollary 4.4(3). $\square$

Corollary 4.7. *Assume that E is an arc length space that is uniformly locally contractible. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points in $X\backslash\{x_0\}$ that converges to $x_0 \in X$. Suppose that $h, k \in U(X,E)$ with $h(x_0) = k(x_0)$, respectively, suppose that $k \in U(X,E)$ and (u_n) is a sequence in E converging to $k(x_0)$. Then there exist infinite sets $I, J \subseteq \mathbb{N}$ and $f \in U(X,E)$ so that*

$$f = \begin{cases} h, \text{ respectively, } u_n & \text{on a neighborhood of } x_n,\ n \in I, \\ k & \text{on a neighborhood of } x_n,\ n \in J. \end{cases}$$

Furthermore, if E is Lipschitz locally contractible, and $h, k \in \mathrm{Lip}(X,E)$, respectively, $\sup \frac{d(u_n,k(x_0))}{d(x_n,x_0)} < \infty$ and $k \in \mathrm{Lip}(X,E)$, then f can be chosen from $\mathrm{Lip}(X,E)$.

Proof. Let ω and M be the parameters for uniform local contractibility of E. We may assume that $\omega_h, \omega_k \le \omega$. Replace x_n by subsequence to assume that $0 < 3d(x_{n+1}, x_0) < d(x_n, x_0)$ for all n. Set $s_n = \frac{d(x_n,x_0)}{16}$. Then

$$d(x_n, x_0) - 8s_n = \frac{1}{2}d(x_n, x_0) > \frac{3}{2}d(x_{n+1}, x_0) = d(x_{n+1}, x_0) + 8s_{n+1}.$$

Hence the sets $B(x_n, 8s_n)$, $n \in \mathbb{N}$, are pairwise disjoint. Since $s_n \to 0$, we may also assume that $r_n := \omega(8s_n) \le M$ for all n. Apply Proposition 4.3 to the sequence (x_{2n}) with $h_n = h|_{B(x_{2n},7s_{2n})}$, respectively, $h_n = u_{2n}$ on $B(x_{2n}, 7s_{2n})$. We obtain $f : X \to E$ so that $f = h$, respectively $f = u_{2n}$, on a neighborhood of x_{2n} and $f = k$ on a neighborhood of x_{2n-1} for all n. Also, $b_n = \omega(\omega(s_{2n})) \to 0$. Moreover,

$$\begin{aligned} a_n = \begin{cases} d(h(x_{2n}), k(x_{2n})) \\ d(u_{2n}, k(x_{2n})) \end{cases} &\le \begin{cases} d(h(x_{2n}), h(x_0)) + d(k(x_0), k(x_{2n})) \\ d(u_{2n}, k(x_0)) + d(k(x_0), k(x_{2n})) \end{cases} \\ &\le \begin{cases} 2\omega(d(x_{2n}, x_0)) \\ d(u_{2n}, k(x_0)) + \omega(d(x_{2n}, x_0)). \end{cases} \end{aligned}$$

Thus $a_n \to 0$. Hence $f \in U(X, E)$ by Corollary 4.4(1).

Finally, in the Lipschitz case, there exists $C < \infty$ so that $\omega(t) \le Ct$ and that $d(u_n, k(x_0)) \le Cd(x_n, x_0)$. From the above $a_n \le 2Cd(x_{2n}, x_0) = 32Cs_{2n}$ in both cases. Thus $(\frac{a_n}{s_{2n}})$ is bounded. By Corollary 4.4(3), $f \in \mathrm{Lip}(X, E)$. $\square$

Fix $v_0 \in E$ and define $|v| = d(v, v_0)$ for any $v \in E$.

Lemma 4.8. *Let E be an arc length space. Assume that $(u_n), (v_n) \subseteq E$ so that $|u_n| \le |v_n|$ and $2|v_{n-1}| < |v_n|$ for any $n \in \mathbb{N}$. There exists a Lipschitz function $g : [0, \infty) \to E$ so that $g(|v_n|) = u_n$ if n is odd and $g(|v_n|) = v_n$ if n is even. Consequently, the function $\xi : E \to E$ given by $\xi(u) = g(|u|)$ is a Lipschitz function that satisfies $\xi(v_n) = u_n$ if n is odd and $\xi(v_n) = v_n$ if n is even.*

Proof. For simplicity, set $w_n = u_n$ if n is odd and $w_n = v_n$ if n is even (including 0). For each $n \in \mathbb{N}$, there exists a 2-Lipschitz path $\gamma_n : [0, a_n] \to E$ so that $\gamma_n(0) = w_{n-1}$ and $\gamma_n(a_n) = w_n$, where $a_n = d(w_{n-1}, w_n)$. Define

$$\alpha_n : [|v_{n-1}|, |v_n|] \to \mathbb{R} \quad \text{by} \quad \alpha_n(t) = \frac{a_n(t - |v_{n-1}|)}{|v_n| - |v_{n-1}|}.$$

Note that

$$0 \le a_n \le |w_{n-1}| + |w_n| \le |v_{n-1}| + |v_n| < 3(|v_n| - |v_{n-1}|).$$

Hence α_n is a nonnegative Lipschitz function with Lipschitz constant < 3 for any $n \in \mathbb{N}$. Define $g : [0, \infty) \to E$ by

$$g(t) = \gamma_n(\alpha_n(t)) \quad \text{if } t \in [|v_{n-1}|, |v_n|].$$

g is well defined since for $t = |v_n|$,

$$\gamma_n(\alpha_n(t)) = \gamma_n(a_n) = w_n = \gamma_{n+1}(0) = \gamma_{n+1}(\alpha_{n+1}(t)).$$

Let us show that g is Lipschitz. Assume that $0 \le t_1 < t_2$. Choose $m \le n$ so that $t_1 \in [|v_{m-1}|, |v_m|]$, $t_2 \in [|v_{n-1}|, |v_n|]$. If $m = n$, then

$$\begin{aligned} d(g(t_1), g(t_2)) &= d(\gamma_m(\alpha_m(t_1)), \gamma_m(\alpha_m(t_2))) \\ &\le 2(|\alpha_m(t_1) - \alpha_m(t_2)|) \le 6|t_1 - t_2|. \end{aligned}$$

On the other hand, if $m < n$, then

$$\begin{aligned} d(g(t_1), g(t_2)) &\le d(g(t_1), g(|v_m|)) + d(g(|v_m|), g(|v_{n-1}|)) \\ &\quad + d(g(|v_{n-1}|), g(t_2)) \\ &= d(g(t_1), g(|v_m|)) + d(w_m, w_{n-1}) \\ &\quad + d(g(|v_{n-1}|), g(t_2)). \end{aligned} \tag{4.4}$$

As above,

$$\begin{aligned} d(g(t_1), g(|v_m|)) &\le 6(|v_m| - t_1) \le 6(t_2 - t_1), \\ d(g(|v_{n-1}|), g(t_2)) &\le 6(t_2 - |v_{n-1}|) \le 6(t_2 - t_1). \end{aligned}$$

Furthermore, if $m = n - 1$, then $d(w_m, w_{n-1}) = 0$. If $m < n - 1$, then

$$d(w_m, w_{n-1}) \le |v_m| + |v_{n-1}| < 4(|v_{n-1}| - |v_m|) \le 4(t_2 - t_1).$$

Therefore, it follows from (4.4) that

$$d(g(t_1), g(t_2)) \le 16(t_2 - t_1).$$

This proves that g is Lipschitz By definition, $g(|v_n|) = w_n$ for any n. The assertions for ξ follow easily. □

Corollary 4.9. *Assume that E is an arc length space that is uniformly locally contractible. Let $(x_n)_{n\in\mathbb{N}}$ be a separated sequence in X. If $h, k \in U(X, E)$, then there exist infinite sets $I, J \subseteq \mathbb{N}$ and $f \in U(X, E)$ so that $f = h$ on a neighborhood of x_n for each $n \in I$ and that $f = k$ on a neighborhood of x_n for each $n \in J$. Moreover, if E is Lipschitz locally contractible and $h, k \in \mathrm{Lip}(X, E)$, then f can be chosen from $\mathrm{Lip}(X, E)$.*

Proof. Assume that E is uniformly locally contractible with parameters ω and M, with $\omega_h, \omega_k \le \omega$. By taking a subsequence, we may assume that $|h(x_n)| \le |k(x_n)|$ for all n. Choose $\eta > 0$ so that $B(x_n, \eta)$, $n \in \mathbb{N}$, are pairwise disjoint.

Assume that $(k(x_n))$ is bounded in E. Then $\sup_n d(h(x_n), k(x_n)) < \infty$. Apply Corollary 4.5(1) with the sequence (x_{2n}). There exists $0 < s < \eta$ and $f \in U(X, E)$ so that $f = h$ on $\bigcup_n B(x_{2n}, s)$ and $f = k$ outside $\bigcup_n B(x_{2n}, \eta)$. In particular, $f = h$ on a neighborhood of x_{2n} and $f = k$ on a neighborhood of x_{2n-1} for all n. Moreover, by the same corollary, f can be chosen from $\mathrm{Lip}(X, E)$ provided $h, k \in \mathrm{Lip}(X, E)$ and E is Lipschitz locally contractible.

Now suppose that $(k(x_n))$ is unbounded. Let $u_n = h(x_n)$ and $v_n = k(x_n)$, so that $|u_n| \le |v_n|$ for all n. By taking a further subsequence, we may assume that $2|v_{n-1}| < |v_n|$ for all n. By Lemma 4.8, there is a Lipschitz function $\xi : E \to E$ such that $\xi(v_n) = u_n$ if n is odd and $\xi(v_n) = v_n$ if n is even. Let $l : X \to E$ be given by $l = \xi \circ k$. Then $l \in U(X, E)$ or $\mathrm{Lip}(X, E)$, with the second alternative holding whenever $k \in \mathrm{Lip}(X, E)$. Furthermore, $l(x_n) = h(x_n)$ if n is odd and $l(x_n) = k(x_n)$ if n is even. Apply Corollary 4.5(1) with the sequence (x_{2n-1}) and functions h and l to obtain $0 < s < \eta$ and $g \in U(X, E)$, respectively, $\mathrm{Lip}(X,E)$, so that $g = h$ on $\bigcup B(x_{2n-1}, s)$ and $g = l$ outside $\bigcup B(x_{2n-1}, \eta)$. Note that $g = h$ on a neighborhood of x_{2n-1} and $g(x_{2n}) = l(x_{2n}) = k(x_{2n})$. Apply Corollary 4.5(1) again with (x_{2n}), k and g to obtain $0 < s' < s$ and $f \in U(X,E)$, respectively, $\mathrm{Lip}(X,E)$, so that $f = k$ on $\bigcup B(x_{2n}, s')$ and $f = g$ outside $\bigcup B(x_{2n}, s)$. Thus $f = h$ on a neighborhood of each x_{2n-1} and $f = k$ on a neighborhood of each x_{2n}. □

5. ⊥-isomorphisms on sets of uniformly continuous and Lipschitz functions

Let X_i, E_i, $i = 1, 2$, be complete metric spaces. To avoid trivialities, we will always assume that $|X_i|, |E_i| > 1$. Let $A(X_i, E_i)$ be either $U(X_i, E_i)$ or $\mathrm{Lip}(X_i, E_i)$. *Furthermore, we assume that E_i is an arc length space that is uniformly locally contractible if $A(X_i, E_i) = U(X_i, E_i)$, respectively, Lipschitz locally contractible if $A(X_i, E_i) = \mathrm{Lip}(X_i, E_i)$.* Let $R \subseteq A(X_1, E_1) \times A(X_2, E_2)$ be a relation so that $\pi_i(R) = A(X_i, E_i)$, $i = 1, 2$. Assume that R is a *⊥-isomorphism*, i.e., R is a $\perp_{h_1,h_2}$-isomorphism for any $(h_1, h_2) \in R$.

Proposition 5.1.

(1) *For any $(h_1, h_2) \in R$, R is a (h_1, h_2)-weakly regular $\perp_{h_1,h_2}$-isomorphism.*
(2) *Let $(h_1, h_2), (k_1, k_2) \in R$. Then $(k_1, k_2) \ll (h_1, h_2)$.*

Proof. (1) Let $(h_1, h_2) \in R$. By assumption, R is a $\perp_{h_1,h_2}$-isomorphism. We show that R is (h_1, h_2)-weakly regular. Assume $x \in U$, where U is an open set in X_1. Since $|E_1| > 1$, there exists $u \in E_1$ such that $u \neq h_1(x)$. Let k be the constant function on X_1 with value u. Obviously $k \in A(X_1, E_1)$. Choose $\eta > 0$ so that $B(x, 2\eta) \subseteq U$. Apply Corollary 4.5(1) with the sequence consisting of the single term x. We obtain $0 < s < \eta$ and $f \in A(X_1, E_1)$ so that $f = k$ on $B(x, s)$ and $f = h_1$ on $(B(x, \eta))^c$. The set $\sigma_{h_1}(f) := \operatorname{int} \overline{[f \neq h_1]}$ is an open neighborhood of x that is contained in $\overline{B(x, \eta)} \subseteq U$. This proves that $A(X_1, E_1)$ is h_1-weakly regular. Similarly, $A(X_2, E_2)$ is h_2-weakly regular.

(2) Suppose that $U \in \mathrm{RO}(X_1)$ and $x \notin \overline{U}$. Choose $\eta > 0$ so that $B(x, 2\eta) \cap U = \emptyset$. Apply Corollary 4.5(1) with the single term x. We obtain $0 < s < \eta$ and $f \in A(X_1, E_1)$ so that $f = k_1$ on $B(x, s)$ and $f = h_1$ on $[B(x, \eta)]^c$. Thus $f = h_1$ on U and $f_1 = k_1$ on $U' := \operatorname{int} \overline{B(x, s)} \in \mathrm{RO}(X_1)$. This verifies Definition 3.1(1). Definition 3.1(3) follows immediately from Corollary 4.9. Items (2) and (4) of Definition 3.1 follows by symmetry. □

Theorem 5.2. *R is a bijective map from $A(X_1, E_1)$ onto $A(X_2, E_2)$. There are a homeomorphism $\varphi : X_1 \to X_2$ and a map $\Phi : X_2 \times E_1 \to E_2$ so that*

$$Rf(y) = \Phi(y, f(\varphi^{-1}(y))) \text{ for any } f \in A(X_1, E_1) \text{ and } y \in X_2. \tag{5.1}$$

Furthermore, $\Phi(y, \cdot) : E_1 \to E_2$ is a bijection for any $y \in X_2$.

Proof. Let $(h_1, h_2) \in R$. By Theorem 2.1 and Proposition 5.1(1), there is a Boolean isomorphism $\theta_{h_1,h_2} : \mathrm{RO}(X_1) \to \mathrm{RO}(X_2)$ such that for any $(f_1, f_2) \in R$ and any $U \in \mathrm{RO}(X_1)$, $f_1 = h_1$ on U if and only if $f_2 = h_2$ on $\theta(U)$. By Proposition 5.1 and [37, Proposition 2.7], $\theta_{h_1,h_2} = \theta_{k_1,k_2}$ for any $(k_1, k_2) \in R$. Denote the common value by θ. If $(f_1, f_2), (g_1, g_2) \in R$ and $f_1 = g_1$, then $f_2 = g_2$ on $\theta(X_1) = X_2$. This shows that the relation R is a function. By symmetry, R is injective. Since $\pi_2(R) = A(X_2, E_2)$, R is onto. Thus R is a bijection from $A(X_1, E_1)$ onto $A(X_2, E_2)$.

Let $(h_1, h_2) \in R$, $x \in X_1$ and $y \in X_2$. Since $|E_i| > 1$, there are constant functions $u \in A(X_1, E_1)$ and $v \in A(X_2, E_2)$ so that $u(x) \neq h_1(x)$ and $v(y) \neq h_2(y)$. By Proposition 5.1(2), $(u, Ru), (R^{-1}v, v) \ll (h_1, h_2)$.

It follows from Theorem 3.2 that there is a homeomorphism $\varphi_{h_1,h_2} : X_1 \to X_2$ associated with (R, h_1, h_2). That is, for any $(f_1, f_2) \in R$ and any $U \in \mathrm{RO}(X_1)$, $f_1 = h_1$ on U if and only if $f_2 = h_2$ on $\varphi_{h_1,h_2}(U)$.

We claim that $\varphi_{k_1,k_2} = \varphi_{h_1,h_2}$ for any $(k_1, k_2) \in R$. Suppose on the contrary that there exists $x \in X_1$ so that $\varphi_{k_1,k_2}(x) \neq \varphi_{h_1,h_2}(x)$. First consider the case where $k_1(x) \neq h_1(x)$. Choose $V \in \mathrm{RO}(X_2)$ so that $y := \varphi_{k_1,k_2}(x) \notin \overline{V}$ and $\varphi_{h_1,h_2}(x) \in V$. As $(k_1, k_2) \ll (h_1, h_2)$ by Proposition 5.1(2), we can apply (2) of Definition 3.1 to obtain $(g_1, g_2) \in R$ and $V' \in \mathrm{RO}(X_2)$ so that $y \in V'$ and that $g_2 = h_2$ on V and $g_2 = k_2$ on V'. Then $g_1 = h_1$ on $\varphi^{-1}_{h_1,h_2}(V) \ni x$ and $g_1 = k_1$ on $\varphi^{-1}_{k_1,k_2}(V') \ni x$. But this means that $h_1(x) = k_1(x)$, contrary to the assumption. Thus $\varphi_{k_1,k_2}(x) = \varphi_{h_1,h_2}(x)$ provided that $h_1(x) \neq k_1(x)$. If $h_1(x) = k_1(x)$, choose $l_1 \in A(X_1, E_1)$ so that $l_1(x) \neq h_1(x)$. By the previous case, $\varphi_{h_1,h_2}(x) = \varphi_{l_1,l_2}(x) = \varphi_{k_1,k_2}(x)$. So the claim is proved.

Denote the common value of all φ_{h_1,h_2} by φ. For any $u \in E_1$, denote the constant function on X_1 with value u by the same symbol. Define $\Phi : X_2 \times E_1 \to E_2$ by $\Phi(y, u) = Ru(y)$. Let $f \in A(X_1, E_1)$ and $x_0 \in X_1$. Set $u = f(x_0)$ and $y_0 = \varphi(x_0)$. If x_0 is an isolated point in X_1, then $U := \{x_0\} \in \mathrm{RO}(X_1)$ and $f = u$ on U. Hence $Rf = Ru$ on $\varphi\{x_0\} = \{y_0\}$. Thus $Rf(y_0) = Ru(y_0) = \Phi(y_0, u)$. Now suppose that x_0 in not an isolated point in X_1. Choose a sequence (x_n) in X_1 with pairwise distinct terms that converges to x_0. By Corollary 4.7, there are infinite sets $I, J \subseteq \mathbb{N}$, sets $U_n \in \mathrm{RO}(X_1)$, $n \in I \cup J$, so that $x_n \in U_n$ if $n \in I \cup J$, and $g \in A_1(X_1, E_1)$ so that $g = f$ on U_n if $n \in I$ and $g = u$ on U_n if $n \in J$. Thus $Rg = Rf$ on $\varphi(U_n) \ni \varphi(x_n)$ if $n \in I$ and $Rg = Ru$ on $\varphi(U_n) \ni \varphi(x_n)$ if $n \in J$. By continuity of φ, Rg, Rf and Ru, we see that $Rf(y_0) = Rg(y_0) = Ru(y_0) = \Phi(y_0, u)$. This completes the proof of (5.1).

Determine $\Psi : X_1 \times E_2 \to E_1$ by $\Psi(x, v) = R^{-1}v(x)$. Applying the preceding argument to R^{-1} shows that $R^{-1}g(x) = \Psi(x, g(\varphi(x)))$ for any $g \in A(X_2, E_2)$ and $x \in E_1$. Easy calculations using $R^{-1}Ru = u$ and $RR^{-1}v = v$ for any $u \in E_1, v \in E_2$ shows that $\Psi(x, \cdot)$ is the inverse of $\Phi(\varphi(x), \cdot)$ for any $x \in E_1$. Thus $\Phi(y, \cdot) : E_1 \to E_2$ is a bijection for any $y \in E_2$. □

6. Characterizing φ and Φ: uniformly continuous functions

In this section, for $i = 1, 2$, let X_i, E_i be complete metric spaces with $|X_i|, |E_i| > 1$. Assume that E_i is an arc length space that is uniformly locally contractible. Fix a $\perp$-isomorphism $R \subseteq U(X_1, E_1) \times U(X_2, E_2)$ so that $\pi_i(R) = U(X_i, E_i)$. By Theorem 5.2, $R : U(X_1, E_1) \to U(X_2, E_2)$ is a bijection; there are a homeomorphism $\varphi : X_1 \to X_2$ and a map $\Phi : X_2 \times E_1 \to E_2$ so that R has the representation (5.1). Furthermore, for each $y \in X_2$, $\Phi(y, \cdot) : E_1 \to E_2$ is a bijection. Denote its inverse by $\Psi(\varphi^{-1}(y), \cdot)$. The aim of this section is to characterize the maps φ and Φ. The following fact will be used repeatedly below. In a complete metric space, every sequence has either a convergent subsequence or a separated subsequence.

Proposition 6.1. *The map $\varphi : X_1 \to X_2$ is a uniform homeomorphism.*

Proof. By symmetry, it suffices to show that φ is uniformly continuous. Suppose otherwise. There are sequences $(x_n), (x'_n)$ in X_1 so that $0 < d(x_n, x'_n) \to 0$ and that there exists $\eta > 0$ with $d(y_n, y'_n) > \eta$ for all n, where $y_n = \varphi(x_n)$ and $y'_n = \varphi(x'_n)$. If either (x_n) or (y_n) has a convergent subsequence, then there exists an infinite set $I \subseteq \mathbb{N}$ so that $(x_n)_{n\in I}$ converges to some $x_0 \in X_1$. This $(x'_n)_{n\in I}$ converges to x_0; hence $(y_n)_{n\in I}$ and $(y'_n)_{n\in I}$ both converge to $\varphi(x_0)$, which is absurd. Similarly, neither (x'_n) nor (y'_n) can have convergent subsequences. Using the observation preceding the proposition and dropping to subsequences, we may assume that (x_n), (x'_n), (y_n) and (y'_n) are all separated sequences. Since $d(y_n, y'_n) > \eta > 0$, we may further assume that $(y_n) \cup (y'_n)$ is separated. To proceed, we consider three cases.

Case 1. There exists $r > 0$ and an infinite set $I \subseteq \mathbb{N}$ so that $B(y_n, r) = \{y_n\}$ for any $n \in I$.

Let u, u' be distinct points in E_1. Define $g : X_2 \to E_2$ by

$$g(y) = \begin{cases} \Phi(y, u') & \text{if } y \notin \{y_n : n \in I\}, \\ \Phi(y_n, u) & \text{if } y = y_n \text{ for some } n \in I. \end{cases}$$

Then $g \in U(X_2, E_2)$ and thus $R^{-1}g \in U(X_1, E_1)$. However, $d(x_n, x'_n) \to 0$ and $R^{-1}g(x_n) = u$, $R^{-1}g(x'_n) = u'$, contrary to the uniform continuity of $R^{-1}g$.

Now suppose that for any $r > 0$, $|\{n : B(y_n, r) = \{y_n\}| < \infty$. By taking a subsequence, we may assume that there exists $(y''_n) \subseteq X_2$ so that $0 < d(y_n, y''_n) \to 0$. Let $x''_n = \varphi^{-1}(y''_n)$.

Case 2. $d(x_n, x''_n) \to 0$.

Pick distinct points $v, v' \in E_2$. Since $(y_n) \cup (y'_n)$ is separated, there exists $\eta > 0$ so that the sets $B(y_n, \eta)$, $n \in \mathbb{N}$, are pairwise disjoint and that $y'_m \notin \bigcup_n B(y_n, \eta)$ for any m. Let k be the constant function with value v'. By Corollary 4.5(2), there exist $0 < s < \eta$ and $g \in U(X_2, E_2)$ so that $g(y_n) = v$ and $g(y'_n) = k(y'_n) = v'$ for all n. Let $u_n = R^{-1}g(x_n)$. Note that $R^{-1}g(x'_n) = R^{-1}k(x'_n)$. Since $0 < d(x_n, x'_n) \to 0$,

$$d(u_n, R^{-1}k(x_n)) \leq d(R^{-1}g(x_n), R^{-1}g(x'_n)) + d(R^{-1}k(x'_n), R^{-1}k(x_n)) \to 0.$$

By Corollary 4.6, there exists $h \in U(X_1, E_1)$ so that $h(x_n) = u_n$ and $h(x''_n) = R^{-1}k(x''_n)$ for infinitely many n. Then $Rh(y_n) = g(y_n) = v$ and $Rh(y''_n) = k(y''_n) = v'$. This is impossible since $Rh \in U(X_2, E_2)$, $d(y_n, y''_n) \to 0$ and $v \neq v'$.

Case 3. $d(x_n, x''_n) \not\to 0$.

Since $(y_n) \cup (y'_n)$ is separated and $d(y_n, y''_n) \to 0$, we may assume that $(y'_n) \cup (y''_n)$ is separated. Thus, without loss of generality, both (x_n) and (x''_n) are separated. With the case assumption, we may assume that $(x_n) \cup (x''_n)$ is separated as well. Let v' and v'' be distinct points in E_2 and let k be the constant function on X_2 with value v'. Since $\sup d(v'', k(y''_n)) = d(v'', v') < \infty$, by Corollary 4.5(2), there exists $g \in U(X_2, E_2)$ so that $g(y''_n) = v''$ and $g(y'_n) = k(y'_n) = v'$ for all n. Then $R^{-1}g(x'_n) = R^{-1}k(x'_n)$. Since $d(x_n, x'_n) \to 0$,

$$\begin{aligned} d(R^{-1}k(x_n), R^{-1}g(x_n)) \leq\ & d(R^{-1}k(x_n), R^{-1}k(x'_n)) \\ & + d(R^{-1}g(x'_n), R^{-1}g(x_n)) \to 0. \end{aligned}$$

By Corollary 4.5, there exists $h \in U(X_1, E_1)$ so that $h(x_n) = R^{-1}k(x_n)$ and $h(x''_n) = R^{-1}g(x''_n)$ for infinitely many n. But then $Rh(y_n) = k(y_n) = v'$ and $Rh(y''_n) = g(y''_n) = v''$, which is impossible since $v' \neq v''$ and $d(y_n, y''_n) \to 0$. $\square$

Denote the set of accumulation points of X_i by X'_i

Proposition 6.2. *The map Φ is continuous at any $(y_0, u_0) \in X'_2 \times E_1$.*

Proof. Let $((y_n, u_n))_n$ be a sequence converging to (y_0, u_0). It suffices to produce a subsequence such that $\Phi(y_{n_k}, u_{n_k}) \to \Phi(y_0, u_0)$. In the first case, assume that $y_n \neq y_0$ for all n. Set $x_n = \varphi^{-1}(y_n)$, $x_0 = \varphi^{-1}(y_0)$. Then (x_n) converges to x_0 and $x_n \neq x_0$ for all n. Resorting to a subsequence, we may assume that (x_n) consists of pairwise distinct points. By Corollary 4.7, there exists $f \in U(X, E)$ and infinite sets $I, J \subseteq \mathbb{N}$ so that $f(x_n) = u_n$ if $n \in I$ and $f(x_n) = u_0$ if $n \in J$. In particular, $f(x_0) = u_0$. Thus

$$Rf(y_n) = \Phi(y_n, f(x_n)) = \Phi(y_n, u_n) \ \text{ if } n \in I \text{ and } \ Rf(y_0) = \Phi(y_0, u_0).$$

Hence $\Phi(y_n, u_n) \to \Phi(y_0, u_0)$ (after taking a subsequence).

In the second case, we may assume that $y_n = y_0$ for all n. Since y_0 is an accumulation point and Ru_n is a continuous function, there is a sequence (y_n') converging to y_0, $y_n' \neq y_0$ for all n, so that $d(Ru_n(y_n), Ru_n(y_n')) \to 0$. By the first case, $Ru_n(y_n') = \Phi(y_n', u_n) \to \Phi(y_0, u_0)$. It follows that $\Phi(y_n, u_n) = Ru_n(y_n) \to \Phi(y_0, u_0)$. $\square$

Given $r > 0$, define $d_r : X_i \times X_i \to [0, \infty]$ by

$$d_r(p,q) = \inf\Big\{\sum_{j=1}^{n} d(x_{j-1}, x_j) : x_j \in X_i, x_0 = p, x_n = q,$$
$$d(x_{j-1}, x_j) \le r, 1 \le j \le n\Big\}.$$

Here we take $\inf \emptyset = \infty$. Note that $d_r(p,q) \ge d(p,q)$, with equality when $d(p,q) \le r$. Call a sequence $((x_n, u_n))_{n=1}^{\infty}$ in $X_1 \times E_1$ a *u-sequence* if (x_n) is separated and there are $r > 0$ and $1 \le C < \infty$ so that

$$d(u_m, u_n) \le C d_r(x_m, x_n) \text{ for any } m, n \in \mathbb{N}.$$

u-sequences in $X_2 \times E_2$ are defined similarly. The next two lemmas give a connection between u-sequences and uniformly continuous functions.

Lemma 6.3. *If $((x_n, u_n))_{n=1}^{\infty}$ is a u-sequence in $X_1 \times E_1$, then there is a function $f \in U(X_1, E_1)$ such that $f(x_n) = u_n$ for infinitely many n.*

Proof. Let r and C be as in the definition of u-sequence. Define an equivalence relation $\sim$ on X_1 by $z_1 \sim z_2$ if $d_r(z_1, z_2) < \infty$. If the sequence (x_n) is not contained in the union of finitely many equivalence classes, then exists an infinite set $I \subseteq \mathbb{N}$ so that $d_r(x_m, x_n) = \infty$ whenever $m, n \in I$, $m \neq n$. Define $f : X_1 \to E_1$ by $f(x) = u_n$ if $x \sim x_n$ for some $n \in I$ and $f(x) = u_1$ otherwise. If $p, q \in X_1$ and $d(p,q) \le r$, then p, q belong to the same equivalence class. Thus $f(p) = f(q)$. Hence $f \in U(X_1, E_1)$. Clearly, $f(x_n) = u_n$ if $n \in I$.

In the second case, the sequence (x_n) is contained in the union of finitely many equivalence classes. By taking a subsequence, we may assume that $x_m \sim x_n$ for all m, n. Let k be the constant function on X_1 with value u_1. If (u_n) is bounded, since (x_n) is separated, it follows from Corollary 4.5(2) that there exists $f \in U(X_1, E_1)$ so that $f(x_n) = u_n$ for any n. Finally, assume that (u_n) is unbounded. Let $|u| = d(u, u_1)$ for any $u \in E_1$. By taking a further subsequence, we may assume that $\max\{2|u_n|, 2Cd_r(x_n, x_1)\} < |u_{n+1}|$ if $n \in \mathbb{N}$. Apply Lemma 4.8 with $v_n = u_n$ to obtain a Lipschitz function $g : [0, \infty) \to E_1$ so that $g(|u_n|) = u_n$ for any n. For any $n \in \mathbb{N}$,

$$2Cd_r(x_n, x_1) < |u_{n+1}| \le Cd_r(x_{n+1}, x_1).$$

Hence $d_r(x_n, x_1) < \frac{1}{2} d_r(x_{n+1}, x_1)$. Thus

$$|u_{n+1}| - |u_n| \le |u_{n+1}| \le 2C(d_r(x_{n+1}, x_1) - d_r(x_n, x_1)).$$

It follows that the function $\alpha : [0, \infty) \to [0, \infty)$ whose graph interpolates the points $(d_r(x_n, x_1), |u_n|)$, $n \in \mathbb{N}$, by straight line segments is a Lipschitz function. Let $X_0 = \{x \in X_1 : x \sim x_1\}$. Define $f : X_1 \to E_1$ by

$$f(x) = \begin{cases} (g \circ \alpha)(d_r(x, x_1)) & \text{if } x \in X_0 \\ u_1 & \text{otherwise.} \end{cases}$$

Assume that $p, q \in X_1$ and that $d(p,q) = c \le r$. Then either p, q both lie in X_0 or both are outside X_0. If $p, q \in X_0$, then

$$|d_r(p,x_1) - d_r(q,x_1)| \le d_r(p,q) = d(p,q) = c.$$

Hence

$$d(f(p), f(q)) = d((g \circ \alpha)(d_r(p,x_1)), (g \circ \alpha)(d_r(q,x_1))) \le \omega_{g\circ\alpha}(c).$$

Otherwise, $p, q \notin X_0$ and $f(p) = u_1 = f(q)$. This proves that $\omega_f \le \omega_{g\circ\alpha}$ on the interval $[0, r]$ and hence f is uniformly continuous. Since $x_n \in X_0$ for all n,

$$f(x_n) = (g \circ \alpha)(d_r(x_n, x_1)) = g(|u_n|) = u_n,$$

as desired. $\square$

Lemma 6.4. *If $f \in U(X_1, E_1)$ and (x_n) is a separated sequence in X_1, then $((x_n, f(x_n)))_{n=1}^\infty$ is a u-sequence.*

Proof. There exists $r > 0$ so that $d(x_m, x_n) > 2r$ if $m \neq n$ and that $d(f(p), f(q)) \le 1$ if $d(p,q) \le r$. Set $u_n = f(x_n)$. Let m, n be distinct points in $\mathbb{N}$. If $d_r(x_m, x_n) = \infty$, then $d(u_m, u_n) \le C d_r(x_m, x_n)$ for any $1 \le C < \infty$. Now assume that $d_r(x_m, x_n) < \infty$. Given $\eta > 0$, choose $a_0 = x_m, \ldots, a_k = x_n$ such that $d(a_{j-1}, a_j) \le r$, $1 \le j \le k$, and that $\sum_{j=1}^k d(a_{j-1}, a_j) \le d_r(x_m, x_n) + \eta$. Since $d(x_m, x_n) > 2r$, $k \ge 2$. By dropping some terms if necessary, we may also assume that $d(a_{j-1}, a_j) + d(a_j, a_{j+1}) > r$ if $1 \le j < k$. Thus

$$\eta + d_r(x_m, x_n) \ge \sum_{j=1}^k d(a_{j-1}, a_j) \ge \frac{(k-1)r}{2}.$$

Taking $\eta \downarrow 0$ gives $d_r(x_m, x_n) \ge \frac{(k-1)r}{2}$. Therefore,

$$d(u_m, u_n) \le \sum_{j=1}^k d(f(a_{j-1}), f(a_j)) \le k \le \frac{2k}{(k-1)r} \cdot d_r(x_m, x_n).$$

It follows that $d(u_m, u_n) \le \frac{4}{r} d_r(x_m, x_n)$. $\square$

For simplicity, let $\Phi \odot \varphi : X_1 \times E_1 \to E_2$ be the function defined by $(\Phi \odot \varphi)(x, u) = \Phi(\varphi(x), u)$. Define $\Psi \odot \varphi^{-1}$ similarly.

Proposition 6.5. *The map $\Phi \odot \varphi : X_1 \times E_1 \to E_2$ is u-*continuous*, i.e., if $((x_n, u_n))_{n=1}^\infty$ is a u-sequence in $X_1 \times E_1$, then for any $\varepsilon > 0$, there exist $\delta > 0$ and $N \in \mathbb{N}$ so that $d(\Phi(\varphi(x), u), \Phi(\varphi(x_n), u_n)) < \varepsilon$ whenever $n \ge N$, $0 < d(x, x_n) < \delta$ and $d(u, u_n) < \delta$.*

Proof. Otherwise, there exist $\varepsilon > 0$, $n_1 < n_2 < \cdots$ and $(x'_k, u'_k) \in X_1 \times E_1$, $k \in \mathbb{N}$, so that $0 < d(x'_k, x_{n_k}) \to 0$, $d(u'_k, u_{n_k}) \to 0$ and that

$$d(\Phi(\varphi(x'_k), u'_k), \Phi(\varphi(x_{n_k}), u_{n_k})) > \varepsilon \text{ for all } k. \tag{6.1}$$

Since $((x_{n_k}, u_{n_k}))_{k=1}^\infty$ is a u-sequence, by Lemma 6.3, there exists $f \in U(X_1, E_1)$ so that $f(x_{n_k}) = u_{n_k}$ for infinitely many k. We may in fact assume that the last equality holds for all k. Note that (x_{n_k}) is separated, $d(x'_k, x_{n_k}) \to 0$ and that

$$d(u'_k, f(x'_k)) \le d(u'_k, u_{n_k}) + d(f(x_{n_k}), f(x'_k)) \to 0.$$

By Corollary 4.6, there exists $g \in U(X_1, E_1)$ so that $g(x'_k) = u'_k$ and $g(x_{n_k}) = f(x_{n_k})$ for infinitely many k. Since $Rg \in U(X_2, E_2)$ and $\varphi : X_1 \to X_2$ is a uniform homeomorphism,

$$\begin{aligned}\liminf d(\Phi(\varphi(x'_k), u'_k), &\Phi(\varphi(x_{n_k}), u_{n_k})) \\ &= \liminf d(Rg(\varphi(x'_k)), Rg(\varphi(x_{n_k}))) = 0,\end{aligned}$$

contrary to (6.1). □

Theorem 6.6. *Let X_i, E_i be complete metric spaces with $|X_i|, |E_i| > 1$. Assume that E_i is an arc length space that is uniformly locally contractible. Let $R \subseteq U(X_1, E_1) \times U(X_2, E_2)$ be a $\perp$-isomorphism so that $\pi_i(R) = U(X_i, E_i)$. Then R has the representation (5.1). Let $\Psi : X_1 \times E_2 \to E_1$ be the map so that $\Psi(x, \cdot)$ is the inverse of $\Phi(\varphi(x), \cdot)$ for any $x \in X_1$. Then*

(1) *$\varphi : X_1 \to X_2$ is a uniform homeomorphism.*
(2) *Φ is continuous on $X'_2 \times E_1$, Ψ is continuous on $X'_1 \times E_2$.*
(3) *$\Phi \odot \varphi$ and $\Psi \odot \varphi^{-1}$ are u-continuous.*

Conversely, if φ, Φ and Ψ satisfy conditions (1)-(3), then (5.1) defines a $\perp$-isomorphism from $U(X_1, E_1)$ onto $U(X_2, E_2)$.

Proof. Given R as described above, then (1)-(3) hold by Propositions 6.1, 6.2 and 6.5 respectively. Conversely, assume that Φ and Ψ satisfy (1)-(3). Define R on $U(X_1, E_1)$ by (5.1). We claim that $Rf \in U(X_2, E_2)$ for any $f \in U(X_1, E_1)$. Assume that the claim has been proven. If $g \in U(X_2, E_2)$, define $Sg : X_1 \to E_1$ by $Sg(x) = \Psi(\varphi(x), g(\varphi(x)))$. By symmetry to the claim, we see that $Sg \in U(X_1, E_1)$ for any $g \in U(X_2, E_2)$. Direct calculation shows that S and R are mutual inverses. Thus R is a bijection from $U(X_1, E_1)$ onto $U(X_2, E_2)$. The formula (5.1) shows that R is a $\perp$-isomorphism.

We will now prove the claim above. Let $f \in U(X_1, E_1)$ and let $(y_n), (y'_n) \subseteq X_2$ so that $d(y_n, y'_n) \to 0$. We need to show that $d(Rf(y_n), Rf(y'_n)) \to 0$. Otherwise, by taking a subsequence and setting $x_n = \varphi^{-1}(y_n)$ and $u_n = f(x_n)$, $u'_n = f(x'_n)$, we may assume that there exists $\varepsilon > 0$ so that

$$d(\Phi(\varphi(x_n), u_n), \Phi(\varphi(x'_n), u'_n)) > \varepsilon \text{ for all } n. \tag{6.2}$$

We may further assume that (x_n) is a sequence of distinct points and that $x_n \neq x'_n$ for all n. If (x_n) has a subsequence convergent to some x_0, then x_0 is an accumulation point and $(u_n), (u'_n)$ both converge to $f(x_0)$. Thus $\varphi(x_0)$ is an accumulation point in X_2. By condition (2), $\Phi(\varphi(x_n), u_n) \to \Phi(\varphi(x_0), f(x_0))$ and $\Phi(\varphi(x'_n), u'_n) \to \Phi(\varphi(x_0), f(x_0))$. Thus (6.2) cannot hold. Otherwise, (x_n) is a separated sequence. Furthermore, as $f \in U(X_1, E_1)$, $((x_n, u_n))_{n=1}^\infty$ is a u-sequence by Lemma 6.4. Since φ is a uniform homeomorphism, $0 < d(x_n, x'_n) \to 0$. Thus $d(u_n, u'_n) \to 0$ as well. By u-continuity of $\Phi \odot \varphi$,

$$d(\Phi(\varphi(x_n), u_n), \Phi(\varphi(x'_n), u'_n)) \to 0,$$

contrary to (6.2). □

6.1. Automatic continuity

In this subsection, we apply Theorem 6.6 to show an automatic continuity result for $\perp$-isomorphisms between sets of uniformly continuous functions. If X, E are metric spaces, $f, g : X \to E$ are uniformly continuous functions and $S \subseteq X$, let

$$d_S(f,g) = \sup\{d(f(x),g(x)) : x \in S\} \in [0,\infty].$$

Theorem 6.7. *Let X_i, E_i be complete metric spaces with $|X_i|, |E_i| > 1$. Assume that E_i is an arc length space that is uniformly locally contractible. Let $R \subseteq U(X_1,E_1) \times U(X_2,E_2)$ be a $\perp$-isomorphism so that $\pi_i(R) = U(X_i,E_i)$ and let $\varphi : X_1 \to X_2$ be the homeomorphism associated with R from Theorem 5.2. Assume that S is a set of accumulation points of X_1. For any $f \in U(X_1,E_1)$ and any $\varepsilon > 0$, there exists $\delta > 0$ so that $d_{\varphi(S)}(Rf, Rg) < \varepsilon$ whenever $g \in U(X_1,E_1)$ and $d_S(f,g) < \delta$.*

Proof. Assume that S is a set of accumulation points of X_1 and $f \in U(X_1,E_1)$. Suppose that there exist $\varepsilon > 0$ and $g_n \in U(X_1,E_1)$ so that $d_S(f,g_n) \to 0$ but $d_{\varphi(S)}(Rf, Rg_n) > \varepsilon$ for all n. Choose $(x_n) \subseteq S$ so that

$$d(Rf(\varphi(x_n)), Rg_n(\varphi(x_n))) > \varepsilon \text{ for all } n. \tag{6.3}$$

By taking a subsequence, we may assume that either (x_n) is a separated sequence or that (x_n) converges in X_1. In the latter case, (x_n) converges to some accumulation point x_0 of X_1. Since $\varphi(x_0)$ is an accumulation point of X_2, by Proposition 6.2, Φ is continuous at $(\varphi(x_0), f(x_0))$. Now

$$\begin{aligned} d(g_n(x_n), f(x_0)) &\le d(g_n(x_n), f(x_n)) + d(f(x_n), f(x_0)) \\ &\le d_S(f,g_n) + d(f(x_n), f(x_0)) \to 0. \end{aligned}$$

Hence

$$Rg_n(\varphi(x_n)) = \Phi(\varphi(x_n), g_n(x_n)) \to \Phi(\varphi(x_0), f(x_0)) = \lim Rf(\varphi(x_n)),$$

contradicting (6.3).

Finally, consider the case where (x_n) is a separated sequence. By Lemma 6.4, $(x_n, f(x_n))$ is a u-sequence. By Theorem 6.6(3), $\Phi \odot \varphi$ is u-continuous. Thus there are $\delta > 0$ and $N \in \mathbb{N}$ so that

$$d(\Phi(\varphi(x), u), \Phi(\varphi(x_n), f(x_n))) < \varepsilon$$

whenever $n \ge N$, $0 < d(x, x_n) < \delta$ and $d(u, f(x_n)) < \delta$. Choose $n \ge N$ so that $d_S(f, g_n) < \delta$. Then $d(f(x_n), g_n(x_n)) < \delta$. By continuity of g_n, there exists $\delta' > 0$ so that $d(f(x_n), g_n(x)) < \delta$ if $d(x, x_n) < \delta'$. If $0 < d(x, x_n) < \min\{\delta, \delta'\}$, and $d(g_n(x), f(x_n)) < \delta$ and hence

$$d(\Phi(\varphi(x), g_n(x)), \Phi(\varphi(x_n), f(x_n))) < \varepsilon$$

by choice of δ and N. Since x_n is an accumulation point and Φ, φ, g_n are continuous functions, upon taking $x \to x_n$

$$d(Rg_n(\varphi(x_n)), Rf(\varphi(x_n))) = d(\Phi(\varphi(x_n), g_n(x_n)), \Phi(\varphi(x_n), f(x_n))) \le \varepsilon,$$

contrary to (6.3). $\square$

7. Characterizing φ and Φ: Lipschitz functions

Let X, E be complete metric spaces with $|X|, |E| > 1$. Assume that E is an arc length space that is Lipschitz locally contractible, with parameters ω and M. The following is an analog of u-sequences in the Lipschitz setting. One of the main goals of this section is to uncover the relationship between L-sequences and Lipschitz functions.

Definition 7.1. A sequence $(x_n, x'_n, u_n, u'_n)_{n=1}^\infty$ in $X^2 \times E^2$ is an *L-sequence* if there exists $C < \infty$ so that for any $m, n \in \mathbb{N}$, $d(u_n, u_m) \le Cd(x_n, x_m)$, $d(u'_n, u'_m) \le Cd(x'_n, x'_m)$ and $d(u_m, u'_n) \le Cd(x_m, x'_n)$.

If $(x_n, x_n, u_n, u_n)_n$ is an L-sequence, we abbreviate it by saying that $(x_n, u_n)_n$ is an L -sequence.

Proposition 7.2. *If $(x_n, u_n)_n$ is an L-sequence in $X \times E$, then there exists $f \in \mathrm{Lip}(X, E)$ so that $f(x_n) = u_n$ for infinitely many n.*

Proof. By dropping to a subsequence if necessary, we may assume that either (x_n) converges to some x_0 or that (x_n) is separated. Let C be the constant from Definition 7.1. If (x_n) converges, then it is Cauchy and hence so is (u_n). Thus (u_n) converges to some u_0. Let k be the constant function on X_1 with value u_0. For any $n \in \mathbb{N}$,

$$d(u_n, k(x_0)) = \lim_m d(u_n, u_m) \le \lim_m Cd(x_n, x_m) = Cd(x_n, x_0).$$

By Corollary 4.7, there exists $f \in \mathrm{Lip}(X, E)$ so that $f(x_n) = u_n$ for infinitely many n.

Now suppose that (x_n) is separated and that (u_n) is bounded. Let $k : X_1 \to E_1$ be a constant function and apply Corollary 4.5(2). We obtain $f \in \mathrm{Lip}(X_1, E_1)$ so that $f(x_n) = u_n$ for all n.

Finally, suppose that (x_n) is separated and that (u_n) is unbounded. Since $Cd(x_n, x_1) \ge |u_n| := d(u_n, u_1)$, $(d(x_n, x_1))_n$ is also unbounded. Take a subsequence to ensure that $2|u_n| < |u_{n+1}|$ and that $2d(x_n, x_1) < d(x_{n+1}, x_1)$ for all n. By Lemma 4.8, taking $v_n = u_n$, we obtain a Lipschitz function $g : [0, \infty) \to E$ so that $g(|u_n|) = u_n$ for all n. Now

$$|u_{n+1}| - |u_n| \le |u_{n+1}| \le Cd(x_{n+1}, x_1) \le 2C(d(x_{n+1}, x_1) - d(x_n, x_1)).$$

It follows that the function $\alpha : [0, \infty) \to [0, \infty)$ that interpolates the points $(d(x_n, x_1), |u_n|)$, $n \in \mathbb{N}$, by straight line segments is Lipschitz. Define $f : X \to E$ by $f(x) = (g \circ \alpha)(d(x, x_1))$. Then $f \in \mathrm{Lip}(X, E)$ and $f(x_n) = g(\alpha(d(x_n, x_1))) = g(|u_n|) = u_n$ for all n. $\square$

Lemma 7.3. *Let $B_n := B(x_n, r_n)$, $n \in \mathbb{N}$, be a sequence of balls in X so that $d(x_m, x_n) \ge 2(r_m + r_n)$ if $m \ne n$. Assume that $u_1 \in E$, $C < \infty$ and for each n, $\gamma_n : [0, r_n] \to E$ is a Lipschitz function with Lipschitz constant at most C so that $\gamma_n(r_n) = u_1$ Define $f : X \to E$ by*

$$f(x) = \begin{cases} \gamma_n(d(x, x_n)) & \text{if } x \in B_n \text{ for some } n, \\ u_1 & \text{otherwise.} \end{cases}$$

Then $f \in \mathrm{Lip}(X, E)$.

Proof. Let $z_1, z_2 \in X$. If $z_1, z_2 \notin \bigcup B_n$, then $f(z_1) = f(z_2)$. If there exists n so that $z_1, z_2 \in B_n$, then $d(f(z_1), f(z_2)) \le Cd(z_1, z_2)$ by choice of f. Suppose that $z_1 \in B_n$ for some n and $z_2 \notin \bigcup_m B_m$. Then

$$d(z_1, z_2) \ge d(z_2, x_n) - d(z_1, x_n) \ge r_n - d(z_1, x_n),$$

while

$$d(f(z_1), f(z_2)) = d(\gamma_n(d(z_1, x_n)), \gamma_n(r_n)) \le C(r_n - d(z_1, x_n)).$$

Finally, suppose that $z_1 \in B_n$ and $z_2 \in B_m$, $m \ne n$. Then

$$\begin{aligned} d(z_1, z_2) &\geq d(x_m, x_n) - d(z_1, x_n) - d(z_2, x_n) \\ &\geq d(x_m, x_n) - r_n - r_m \geq r_m + r_n. \end{aligned}$$

On the other hand, since $\gamma_n(r_n) = u_1 = \gamma_m(r_m)$,

$$\begin{aligned} d(f(z_1), f(z_2)) &\leq d(\gamma_n(d(z_1, x_n)), \gamma_n(r_n)) + d(\gamma_m(r_m), \gamma_m(d(z_2, x_m))) \\ &\leq C(r_n - d(z_1, x_n) + r_m - d(z_2, x_m)) \\ &\leq C(r_n + r_m) \leq Cd(z_1, z_2). \end{aligned}$$

Thus $f \in \mathrm{Lip}(X, E)$. $\square$

Proposition 7.4. *Assume that $(x_n) \cup (x'_n)$ is a separated set in X so that $d(x_n, x_1), d(x'_n, x_1) \to \infty$. If $(x_n, x'_n, u_n, u'_n)_{n=1}^{\infty}$ is an L-sequence, then there exists $f \in \mathrm{Lip}(X, E)$ and an infinite set $J \subseteq \mathbb{N}$ so that $f(x_n) = u_n$, $f(x'_n) = u'_n$ for all $n \in J$.*

Proof. Let C be as in Definition 7.1. By taking a subsequence, we may also assume that $0 < d(x'_n, x_1) \leq d(x_n, x_1) < \frac{1}{2}d(x'_{n+1}, x_1)$ for all n and that $a := \lim_n \frac{d(x_n, x'_n)}{d(x_n, x_1)}$ exists (possibly infinite).

Case 1. $a > 0$.

Let $r_n = \frac{a\wedge 1}{9}d(x_n, x_1)$ and $r'_n = \frac{a\wedge 1}{9}d(x'_n, x_1)$. We may assume that

$$d(x_n, x'_n) > \frac{(a \wedge 1)d(x_n, x_1)}{2} > 4r_n \geq 2(r_n + r'_n) \text{ for all } n.$$

We claim that

$$d(x_m, x_n) \geq 2(r_n + r_m), \quad d(x'_m, x'_n) \geq 2(r'_m + r'_n) \quad \text{if } m \neq n, \text{ and} \tag{7.1}$$

$$d(x_m, x'_n) \geq 2(r_m + r'_n) \text{ for all } m, n. \tag{7.2}$$

Let us prove the first inequality in (7.1). The rest are similar. Without loss of generality, suppose that $n > m$. Then

$$\begin{aligned} d(x_m, x_n) &\geq d(x_n, x_1) - d(x_m, x_1) \\ &\geq \frac{d(x_n, x_1) + d(x_m, x_1)}{4} > 2(r_n + r_m). \end{aligned}$$

Note that there is a constant $C' < \infty$ so that $d(u_n, u_1) \leq C'r_n$ and $d(u'_n, u_1) \leq C'r'_n$. Since E is an arc length space, for each n, there are $2C'$-Lipschitz functions $\gamma_n : [0, r_n] \to E$, $\gamma'_n : [0, r'_n] \to E$ so that

$$\gamma_n(0) = u_n, \gamma'_n(0) = u'_n, \gamma_n(r_n) = \gamma'_n(r'_n) = u_1.$$

Set $B_n = B(x_n, r_n)$ and $B'_n = B(x'_n, r'_n)$. By (7.1) and (7.2), Lemma 7.3 applies to the sequence of balls $(B_n)\bigcup(B'_n)$ and the functions $(\gamma_n)\bigcup(\gamma'_n)$. Hence we obtain a Lipschitz function $f : X \to E$ so that $f(x_n) = \gamma_n(0) = u_n$ and $f(x'_n) = \gamma'_n(0) = u'_n$ for all n.

Case 2. $a = 0$.

We may assume that $d(x_n, x'_n) < \frac{1}{10}d(x_n, x_1)$ for all n. Let $r_n = \frac{d(x_n, x_1)}{9}$ and $B_n = B(x_n, r_n)$. As in Case 1, we find that $d(x_m, x_n) \geq 2(r_n + r_m)$ if $m \neq n$. Also,

$$\begin{aligned} d(u_n, u_n') &\le Cd(x_n, x_n'),\\ d(u_n', u_1) &\le Cd(x_n', x_1) \le C(d(x_n', x_n) + d(x_n, x_1))\\ &\le 2Cd(x_n, x_1) = 18Cr_n\\ &\le 180C(r_n - d(x_n, x_n')). \end{aligned}$$

Since E is an arc length space, there are paths

$$\begin{aligned} &\gamma_n' : [0, d(x_n, x_n')] \to E \text{ that goes from } u_n \text{ to } u_n' \text{ and}\\ &\gamma_n'' : [d(x_n, x_n'), r_n] \to E \text{ that goes from } u_n' \text{ to } u_1, \end{aligned}$$

with $\mathrm{Lip}(\gamma') \le 2C$ and $\mathrm{Lip}(\gamma'') \le 360C$. Concatenate these curves to obtain a $360C$-Lipschitz curve $\gamma_n; [0, r_n] \to E$ so that $\gamma_n(0) = u_n$, $\gamma_n(d(x_n, x_n')) = u_n'$ and that $\gamma(r_n) = u_1$. By Lemma 7.3, we obtain a function $f : X \to E$ so that $f(x_n) = \gamma_n(0) = u_n$ and $f(x_n') = \gamma_n(d(x_n', x_n)) = u_n'$. $\square$

We now arrive at the main characterization theorem for L-sequences.

Proposition 7.5. *A sequence $(x_n, x_n', u_n, u_n')_{n=1}^\infty$ in $X^2 \times E^2$ contains an L-subsequence if and only if there exist $f \in \mathrm{Lip}(X, E)$ and an infinite set $I \subseteq \mathbb{N}$ so that $f(x_n) = u_n$ and $f(x_n') = u_n'$ for all $n \in I$.*

Proof. The "if" case is clear. Conversely, we may assume that $(x_n, x_n', u_n, u_n')_{n=1}^\infty$ is an L-sequence. Since $(x_n, u_n)_n$ is an L-sequence, by Proposition 7.2, there are an infinite set $I \subseteq \mathbb{N}$ and $g \in \mathrm{Lip}(X, E)$ so that $g(x_n) = u_n$ for all $n \in I$. Similarly, there exist an infinite set $J \subseteq I$ and $h \in \mathrm{Lip}(X, E)$ so that $h(x_n') = u_n'$ for all $n \in J$. After relabeling, we may assume that $g(x_n) = u_n$ and $h(x_n') = u_n'$ for all $n \in \mathbb{N}$. By taking a further subsequence, we may also assume that each of (x_n) and (x_n') is either convergent or separated. Let C be the constant from Definition 7.1. Without loss of generality, suppose that $\omega_g(t), \omega_h(t), \omega(t) \le Ct$. We divide the rest of the proof into several cases.

<u>Case 1</u>. (x_n) converges to x_0 and (x_n') does not converge to x_0.

Choose $\eta > 0$ and an infinite set $I \subseteq \mathbb{N}$ so that $x_n' \notin B(x_0, \eta)$ for all $n \in I$. Apply Corollary 4.5(1) with the single point x_0 and the functions g and h. We obtain $0 < s < \eta$ and $f \in \mathrm{Lip}(X, E)$ so that $f = g$ on $B(x_0, s)$ and $f(x_n') = h(x_n')$ for all $n \in I$. Hence there is an infinite set $J \subseteq I$ so that $f(x_n) = g(x_n) = u_n$ and $f(x_n') = h(x_n') = u_n'$ for all $n \in J$.

Of course, the same argument applies if (x_n') converges to x_0 but (x_n) does not.

<u>Case 2</u>. (x_n) and (x_n') both converge to x_0.

If either $x_n' = x_n$ or $x_n' = x_0$ for infinitely many n, then the result follows from Proposition 7.2. Otherwise, by taking a further subsequence, we may assume that $x_n' \neq x_n, x_0$ for all n and that $d(x_n, x_0) \ge d(x_n', x_0) > 3d(x_{n+1}, x_0)$ for all n. Note that

$$d(x_n, x_n') \le d(x_n, x_0) + d(x_n', x_0) \le 2d(x_n, x_0).$$

Set $s_n = \frac{d(x_n, x_n')}{32}$, so that $8s_n \le \frac{1}{2}d(x_n, x_0)$. Then it is easy to check that $B(x_n, 8s_n)$, $n \in \mathbb{N}$, are pairwise disjoint and that $x_m' \notin \bigcup_n B(x_n, 6s_n)$ for any m. Since (s_n) is a null sequence, we may also assume that $\omega(8s_n) \le M$ for all n. Apply Proposition 4.3 with $h_n = g|_{B(x_n, 7s_n)}$ and $k = h$. We obtain $f : X \to E$ so that $f(x_n) = g(x_n) = u_n$ and $f(x_n') = k(x_n') = u_n'$ for all n. Then

$$d(g(x_n), h(x_n)) \le d(u_n, u_n') + d(h(x_n'), h(x_n)) \le 2Cd(x_n, x_n') = 64Cs_n.$$

In the notation of Proposition 4.3, $(\frac{a_n}{s_n})$ is bounded. Thus $f \in \mathrm{Lip}(X, E)$ by Corollary 4.4(3).

Case 3. (x_n) and (x'_n) are both separated and $d(x_n, x'_n) \to 0$.

Choose $\eta > 0$ so that $B(x_n, \eta)$ are pairwise disjoint. We may assume that $8d(x'_n, x_n) < \eta$ for all n. Set $s_n = \frac{d(x_n,x'_n)}{8}$. Then $\omega(s_n) \leq M$ for all sufficiently large n, $B(x_n, 8s_n)$, $n \in \mathbb{N}$, are pairwise disjoint and $x'_m \notin \bigcup_n B(x_n, 6s_n)$ for any m. Apply Proposition 4.3 with $h_n = g|_{B(x_n,7s_n)}$ and $k = h$ to obtain $f : X \to E$ so that $f(x_n) = g(x_n) = u_n$ and $f(x'_n) = k(x'_n) = h(x'_n) = u'_n$ for all n. In the notation of Proposition 4.3,

$$a_n = d(g(x_n), h(x_n)) \leq d(u_n, u'_n) + d(h(x'_n), h(x_n)) \leq 2Cd(x_n, x'_n).$$

Hence $(\frac{a_n}{s_n})$ is bounded. Thus $f \in \mathrm{Lip}(X, E)$ by Corollary 4.4(3).

Case 4 . (x_n) and (x'_n) are both separated and $d(x_n, x'_n) \not\to 0$.

In this case, we may assume that $(x_n) \cup (x'_n)$ is separated. If (x_n) has a bounded subsequence, then we may assume that the sequences $(g(x_n))$ and $(h(x_n))$ are bounded. Choose $\eta > 0$ so that the sets $B(x_n, \eta)$, $n \in \mathbb{N}$, are disjoint and that $x'_m \notin \bigcup_n B(x_n, \eta)$ for any m. By Corollary 4.5(1), there exist $f \in \mathrm{Lip}(X, E)$ so that $f(x_n) = g(x_n) = u_n$ for each n and that $f = h$ outside $\bigcup_n B(x_n, \eta)$. Thus $f(x'_n) = h(x'_n) = u'_n$ for each n. The result follows by symmetry if (x'_n) has a bounded subsequence. In the final case, assume that neither (x_n) nor (x'_n) have bounded subsequences. By taking a further subsequence, we may assume that $(x_n) \cup (x'_n)$ is separated and that $d(x_n, x_1), d(x'_n, x_1) \to \infty$. The desired conclusion follows from Proposition 7.4. □

Let X_i, E_i be complete metric spaces so that $|X_i|, |E_i| > 1$, $i = 1, 2$. Assume that E_i is an arc length space that is Lipschitz locally contractible. Fix a $\perp$-isomorphism $R \subseteq \mathrm{Lip}(X_1, E_1) \times \mathrm{Lip}(X_2, E_2)$ so that $\pi_i(R) = \mathrm{Lip}(X_i, E_i)$, $i = 1, 2$. By Theorem 5.2, R is a bijection from $\mathrm{Lip}(X_1, E_1)$ onto $\mathrm{Lip}(X_2, E_2)$; there are a homeomorphism $\varphi : X_1 \to X_2$ and a map $\Phi : X_2 \times E_1 \to E_2$ so that R has the representation (5.1). Furthermore, for each $y \in X_2$, $\Phi(y, \cdot) : E_1 \to E_2$ is a bijection. Denote its inverse by $\Psi(\varphi^{-1}(y), \cdot)$. Let $(\varphi \cdot \Phi) : X_1^2 \times E_1^2 \to X_2^2 \times E_2^2$ be the map

$$(\varphi \cdot \Phi)(x, x', u, u') = (\varphi(x), \varphi(x'), \Phi(\varphi(x), u), \Phi(\varphi(x'), u')).$$

The map $\varphi \cdot \Phi$ is an *L-preserver* if $(x_n, x'_n, u_n, u_n)_n$ in $X_1^2 \times E_1^2$ contains an L-subsequence if and only if $((\varphi \cdot \Phi)(x_n, x'_n, u_n, u'_n))_n$ contains an L-subsequence.

Theorem 7.6. *Let $R \subseteq \mathrm{Lip}(X_1, E_1) \times \mathrm{Lip}(X_2, E_2)$ be a $\perp$-isomorphism so that $\pi_i(R) = \mathrm{Lip}(X_i, E_i)$, $i = 1, 2$. In the notation above, $\varphi \cdot \Phi$ is an L-preserver.*

Conversely, suppose that $\varphi : X_1 \to X_2$ is a homeomorphism; $\Phi : X_2 \times E_1 \to E_2$ is a map so that $\Phi(y, \cdot) : E_1 \to E_2$ is a bijection for each $y \in X_2$ and that $\varphi \cdot \Phi$ is an L-preserver. Define R by (5.1). Then R is a $\perp$-isomorphism from $\mathrm{Lip}(X_1, E_1)$ onto $\mathrm{Lip}(X_2, E_2)$.

Proof. Let $R \subseteq \mathrm{Lip}(X_1, E_1) \times \mathrm{Lip}(X_2, E_2)$ be a $\perp$-isomorphism so that $\pi_i(R) = \mathrm{Lip}(X_i, E_i)$, $i = 1, 2$. Determine φ and Φ as in the preceding paragraph. Suppose that $(x_n, x'_n, u_n, u_n)_n$ contains an L-subsequence. By Proposition 7.5, there exists $f \in \mathrm{Lip}(X_1, E_1)$ and an infinite set $I \subseteq \mathbb{N}$ so that $f(x_n) = u_n$ and $f(x'_n) = u'_n$ for all $n \in I$. Then $Rf(\varphi(x_n)) = \Phi(\varphi(x_n), f(x_n)) = \Phi(\varphi(x_n), u_n)$ and $Rf(\varphi(x'_n)) = \Phi(\varphi(x'_n), u'_n)$ for all $n \in I$. Since $Rf \in \mathrm{Lip}(X_2, E_2)$,

$$((\varphi \cdot \Phi)(x_n, x'_n, u_n, u'_n))_{n\in I} = (\varphi(x_n), \varphi(x'_n), \Phi(\varphi(x_n), u_n), \Phi(\varphi(x'_n), u'_n))_{n\in I}$$

is an L-sequence. A similar argument shows that $(x_n, x'_n, u_n, u'_n)_n$ contains an L-subsequence if $((\varphi \cdot \Phi)(x_n, x'_n, u_n, u'_n))_n$ contains an L-subsequence.

Conversely, assume that φ and Φ are as given in the statement of Theorem 7.6. For any $f \in \mathrm{Lip}(X_1, E_1)$, define $Rf : X_2 \to E_2$ by $Rf(y) = \Phi(y, f(\varphi^{-1}(y)))$. If Rf fails to be Lipschitz, there are sequences $(y_n), (y'_n)$

so that $d(Rf(y_n), Rf(y'_n)) > nd(y_n, y'_n)$ for all n. Let $x_n = \varphi^{-1}(y_n)$, $x'_n = \varphi^{-1}(y'_n)$, $u_n = f(x_n)$ and $u'_n = f(x'_n)$. Then $(x_n, x'_n, u_n, u'_n)_n$ is an L-sequence. Note that $Rf(y_n) = \Phi(y_n, u_n)$ and $Rf(y'_n) = \Phi(y'_n, u'_n)$. By assumption, $(y_n, y'_n, Rf(y_n), Rf(y'_n))$ has an L -subsequence. Without loss of generality, there exists $C < \infty$ so that $d(Rf(y_n), Rf(y'_n)) \leq Cd(y_n, y'_n)$ for all $n \in \mathbb{N}$, contrary to the choices of (y_n) and (y'_n). This shows that $Rf \in \mathrm{Lip}(X_2, E_2)$. If $g \in \mathrm{Lip}(X_2, E_2)$, then let $f : X_1 \to E_1$ be given by $f(x) = \Psi(x, g(\varphi(x)))$, where $\Psi(x, \cdot)$ is the inverse of $\Phi(\varphi(x), \cdot)$. By a similar argument to the above, $f \in \mathrm{Lip}(X_1, E_1)$. Clearly $Rf = g$. Thus $R : \mathrm{Lip}(X_1, E_1) \to \mathrm{Lip}(X_2, E_2)$ is onto. It follows easily from the properties of φ and Φ that R is a $\perp$-isomorphism. $\square$

8. Further properties of φ and Φ

8.1. Lipschitzness of φ

Guided by analogy with the case of uniformly continuous functions, one might conjecture that φ is a Lipschitz homeomorphism. However, this turns out to be true only in the case where both X_1 and X_2 are given bounded metrics. In fact, we will see that φ is a Lipschitz homeomorphism with respect to certain metrics that are topologically equivalent to the original metrics on X_i. The modified metrics are suggested by [35]. In turn, they trace their ancestry to [40].

Two metrics on a set X are *topologically equivalent* if they generate the same topology on X. If E is a bounded metric space, then $\mathrm{Lip}(X, E) = \mathrm{Lip}(X', E)$ as sets, where X' is the set X with the metric $d \wedge 1$. Here d is the original metric on X and $d \wedge 1$ is topologically equivalent to d. With this in mind, *we will always assume additionally that the metric on X is bounded whenever the metric on E is bounded.*

Suppose that X is a complete metric space with a distinguished point x_0. Let $|p| = d(p, x_0)$ for any $x \in X$. Define $\rho : X \times X \to \mathbb{R}$ by

$$\rho(p, q) = \frac{d(p, q)}{|p| \vee |q| \vee 1}.$$

Proposition 8.1. *[35, Proposition 5.1] There is a complete bounded metric d' on X that is topologically equivalent to d and that $\rho(p, q) \leq d'(p, q) \leq 3\rho(p, q)$ for any $p, q \in X$.*

In this subsection, assume that X_i, E_i, $i = 1, 2$, are complete metric spaces with $|X_i|, |E_i| > 1$. Assume that E_i is an arc length space that is Lipschitz locally contractible. Choose distinguished points $x_0 \in X_1$ and $y_0 \in X_2$ and define ρ on $X_i \times X_i$ with respect to the respective distinguished points. (We use the same symbol ρ for $i = 1, 2$.) Let $R \subseteq \mathrm{Lip}(X_1, E_1) \times \mathrm{Lip}(X_2, E_2)$ be a $\perp$-isomorphism so that $\pi_i(R) = \mathrm{Lip}(X_i, E_i)$, $i = 1, 2$. Determine functions $\varphi : X_1 \to X_2$ and $\Phi : X_2 \times E_1 \to E_2$ as in Theorem 7.6. The aim of this subsection is to prove that $\varphi : X_1 \to X_2$ is a ρ-to-ρ homeomorphism.

Theorem 8.2. *The map φ is a Lipschitz homeomorphism from (X_1, ρ) onto (X_2, ρ).*

The proof of Theorem 8.2 contains several cases, which are discussed in the propositions below.

Lemma 8.3. *Let (x_n) be a ρ-separated sequence in X_1 and let $u \in E_1$. Then there exist $f \in \mathrm{Lip}(X_1, E_1)$ and $c_2 > c_1 > 0$ so that $c_1|x_n| \leq d(f(x_n), u) \leq c_2|x_n|$ for infinitely many n.*

Proof. We may assume that $\inf |x_n| > 0$. Since E_1 is path connected and we assume that E_1 is unbounded if X_1 is unbounded, there exist $c_2 > c_1 > 0$ and a sequence (u_n) in E_1 so that $c_1|x_n| \leq d(u_n, u) \leq c_2|x_n|$. Let $\eta > 0$ be such that $\rho(x_m, x_n) > \eta$ if $m \neq n$. Set $r_n = \frac{\eta|x_n|}{5}$ for each n. Then $r_n > 0$ and

$$2(r_n + r_m) \le \frac{4\eta}{5}(|x_n| \vee |x_m| \vee 1) \le \frac{4}{5}d(x_n, x_m).$$

Furthermore, as $d(u_n, u) \le \frac{5c_2 r_n}{\eta}$ for all n, there exist $C < \infty$ and Lipschitz functions $\gamma_n : [0, r_n] \to E_1$ with Lipschitz constants at most C so that $\gamma_n(0) = u_n$ and $\gamma_n(r_n) = u$. By Lemma 7.3, there exists $f \in \mathrm{Lip}(X_1.E_1)$ so that $f(x_n) = \gamma_n(0) = u_n$ for all n. $\square$

Proposition 8.4. *Let (x_n), (x'_n) be sequences in X_1 and let $y_n = \varphi(x_n)$, $y'_n = \varphi(x'_n)$. If (y_n) and (y'_n) are ρ-separated and $\inf \rho(y_n, y'_n) > 0$, then $(\frac{\rho(y_n,y'_n)}{\rho(x_n,x'_n)})$ is bounded.*

Proof. Since the assumptions pass to subsequences, it suffices to show that $(\frac{\rho(y_n,y'_n)}{\rho(x_n,x'_n)})$ has a bounded subsequence. By taking subsequences, we may assume that the set $(y_n) \cup (y'_n)$ is ρ-separated and that $|x_n| \ge |x'_n|$ for all n. In this case, (x_n) cannot have a convergent subsequence and hence we may also assume that it is ρ-separated. Fix $u \in E_1$. By Lemma 8.3, we may assume that there exist $f_1 \in \mathrm{Lip}(X_1, E_1)$ and $c_2 > c_1 > 0$ so that $c_1|x_n| \le d(f_1(x_n), u) \le c_2|x_n|$. Let $f_2 : X_1 \to E_1$ be a constant function with value u. Set $v_n = Rf_1(y_n)$ and $v'_n = Rf_2(y'_n)$ for all n.

We claim that the sequence $(y_n, y'_n, v_n, v'_n)_n$ is an L-sequence. Indeed, since Rf_1 and Rf_2 are Lipschitz functions, $(y_n, v_n)_n$ and $(y'_n, v'_n)_n$ are L-sequences. Let

$$C = \mathrm{Lip}(Rf_1) + d(Rf_1(y_0), Rf_2(y_0)) + \mathrm{Lip}(Rf_2).$$

For any $m, n \in \mathbb{N}$,

$$\begin{aligned} d(v_m, v'_n) &\le d(Rf_1(y_m), Rf_1(y_0)) + d(Rf_1(y_0), Rf_2(y_0)) \\ &\quad + d(Rf_2(y_0), Rf_2(y'_n)) \\ &\le \mathrm{Lip}(Rf_1)|y_m| + d(Rf_1(y_0), Rf_2(y_0)) + \mathrm{Lip}(Rf_2)|y'_n| \\ &\le C(|y_m| \vee |y'_n| \vee 1). \end{aligned}$$

However, there exists $\eta > 0$ such that $\rho(y_m, y'_n) \ge \eta$ for any m, n. Thus

$$d(v_m, v'_n) \le C(|y_m| \vee |y'_n| \vee 1) \le \frac{C}{\eta}d(y_m, y'_n).$$

This completes the proof of the claim.

From the claim and Proposition 7.5, we may assume that there exists $g \in \mathrm{Lip}(X_2, E_2)$ so that $g(y_n) = v_n = Rf_1(y_n)$ and $g(y'_n) = v'_n = Rf_2(y'_n)$ for all n. Now $h = R^{-1}g \in \mathrm{Lip}(X_1, E_1)$ and $h(x_n) = f_1(x_n)$, $h(x'_n) = f_2(x'_n) = u$ for all n. Thus

$$c_1|x_n| \le d(f_1(x_n), u) = d(h(x_n), h(x'_n)) \le \mathrm{Lip}(h)d(x_n, x'_n). \tag{8.1}$$

Since (x_n) is ρ-separated, we may assume that $\inf |x_n| = r > 0$. Then $|x_n| \vee |x'_n| \vee 1 = |x_n| \vee 1 \le |x_n|(1 \vee \frac{1}{r})$. So (8.1) implies that $\rho(x_n, x'_n) \ge \frac{c_1}{\mathrm{Lip}(h)(1 \vee r^{-1})}$. Since ρ is a bounded metric on X_2, it follows that $(\frac{\rho(y_n,y'_n)}{\rho(x_n,x'_n)})$ is bounded. $\square$

Lemma 8.5. *Assume that E_2 is an arc length space. Let $y \in X_2$, $v, \widetilde{v} \in E_2$, $x = \varphi^{-1}(y)$ and let $u, \widetilde{u} \in E_1$ be such that $\Phi(y, u) = v$, $\Phi(y, \widetilde{u}) = \widetilde{v}$. Suppose that $k \in \mathbb{N}$. Then there exist $a_1, a_2 \in E_1$ so that*

$$d(a_1, a_2) \ge \frac{d(u, \widetilde{u})}{k}, \quad d(b_1, b_2) \le \frac{2d(v, \widetilde{v})}{k} \quad \text{and} \quad d(b_i, v) \le 2d(v, \widetilde{v}),$$

where $b_i = \Phi(y, a_i)$, $i = 1, 2$.

Proof. Let $t = d(v, \widetilde{v})$. Since E is an arc length space there exists a 2-Lipschitz path $\gamma : [0, t] \to E_2$ so that $\gamma(0) = v$ and $\gamma(t) = \widetilde{v}$. Define $v_j = \gamma(\frac{jt}{k})$, $0 \leq j \leq k$, and determine $u_j \in E_1$ by $\Phi(y, u_j) = v_j$. Since $v_0 = v$ and $v_k = \widetilde{v}$, $u_0 = u$ and $u_k = \widetilde{u}$. Thus

$$d(u, \widetilde{u}) \leq \sum_{j=1}^{k} d(u_{j-1}, u_j).$$

Hence there exists j_0 so that $d(u_{j_0-1}, u_{j_0}) \geq \frac{d(u,\widetilde{u})}{k}$. On the other hand

$$d(v_{j_0-1}, v_{j_0}) \leq \mathrm{Lip}(\gamma) \cdot \frac{t}{k} \leq \frac{2d(v, \widetilde{v})}{k}.$$

Moreover, $d(v_{j_0-1}, v), d(v_{j_0}, v) \leq 2t$. The lemma is proved by taking $a_1 = u_{j_0-1}$ and $a_2 = u_{j_0}$. $\square$

Proposition 8.6. *Let (x_n), (x'_n) be sequences in X_1, $x_n \neq x'_n$ for all n, and let $y_n = \varphi(x_n)$, $y'_n = \varphi(x'_n)$. If (y_n) and (y'_n) are ρ-separated and $\rho(y_n, y'_n) \to 0$, then $(\frac{\rho(y_n,y'_n)}{\rho(x_n,x'_n)})$ is bounded.*

Proof. As before, it suffices to show that $(\frac{\rho(y_n,y'_n)}{\rho(x_n,x'_n)})$ has a bounded subsequence. Since φ is a homeomorphism, neither (x_n) nor (x'_n) can have convergent subsequences. Hence we may assume that there exists $r > 0$ so that $|x_n| \geq |x'_n| \vee r$, $|y_n|, |y'_n| \geq r$ and that $r \leq \rho(y_n, y_m), \rho(y'_n, y'_m), \rho(y_n, y'_m)$ whenever $n \neq m$. Moreover,

$$\frac{d(y_n, y'_n)}{|y_n| \vee |y'_n| \vee 1} = \rho(y_n, y'_n) \to 0.$$

Thus $\frac{|y_n|}{|y'_n|} \to 1$. Thus we may also assume that $|y_n| \leq 2|y'_n|$ for all n. Let $u \in E_1$. Apply Lemma 8.3 and take a subsequence to find $f \in \mathrm{Lip}(X_1, E_1)$ and $c_2 > c_1 > 0$ so that $c_1|x_n| \leq d(f(x_n), u) \leq c_2|x_n|$ for all n. Define $\widetilde{u_n} = f(x_n)$, $v_n = \Phi(y_n, u)$ and $\widetilde{v_n} = \Phi(y_n, \widetilde{u_n})$. Let g be the constant function with value u.

Case 1. $d(v_n, \widetilde{v_n}) > d(y_n, y'_n)$ for all n.

Let $k_n \in \mathbb{N}$ be the smallest number such that $k_n d(y_n, y'_n) \geq d(v_n, \widetilde{v_n})$. By the case assumption, $k_n \geq 2$ and thus $\frac{k_n}{2} d(y_n, y'_n) < d(v_n, \widetilde{v_n})$. By Lemma 8.5, for each n, there are $a_1^n, a_2^n \in E_1$ with

$$d(a_1^n, a_2^n) \geq \frac{d(u, \widetilde{u_n})}{k_n}, \ d(b_1^n, b_2^n) \leq \frac{2d(v_n, \widetilde{v_n})}{k_n} \ \text{ and } \ d(b_i^n, v_n) \leq 2d(\widetilde{v_n}, v_n),$$

$i = 1, 2$, where $b_i^n = \Phi(y_n, a_i^n)$.

Claim. The sequences $(y_n, y'_n, b_1^n, b_1^n)_n$ and $(y_n, y'_n, b_2^n, b_1^n)_n$ are L-sequences.

For any $m, n \in \mathbb{N}$ and $i, j = 1, 2$,

$$\begin{aligned} d(b_i^n, b_j^m) &\leq d(b_i^n, v_n) + d(v_n, v_m) + d(b_j^m, v_m) \\ &\leq 2d(\widetilde{v_n}, v_n) + d(v_n, v_m) + 2d(v_m, \widetilde{v_m}). \end{aligned}$$

Since $v_n = \Phi(y_n, u) = Rg(y_n)$,

$$d(v_n, v_m) \leq \mathrm{Lip}(Rg)\, d(y_n, y_m) \leq \mathrm{Lip}(Rg)(|y_n| + |y_m|)$$

for any m, n. Also,

$$\begin{aligned} d(\widetilde{v_n}, v_n) &= d(Rf(y_n), Rg(y_n)) \\ &\leq (\mathrm{Lip}(Rf) + \mathrm{Lip}(Rg))|y_n| + d(Rf(y_0), Rg(y_0)). \end{aligned} \tag{8.2}$$

So there exists $K < \infty$ so that

$$d(b_i^n, b_j^m) \le K(|y_n| \vee |y_m| \vee 1),\ i = 1,2,\ m,n \in \mathbb{N}. \tag{8.3}$$

From (8.3), we also obtain

$$d(b_i^n, b_j^m) \le 2K(|y_n| \vee |y'_m| \vee 1), 2K(|y'_n| \vee |y'_m| \vee 1),\ i = 1,2,\ m,n \in \mathbb{N}. \tag{8.4}$$

On the other hand, if $n \neq m$, $r \le \rho(y_n, y_m), \rho(y'_n, y'_m), \rho(y_n, y'_m)$. That is,

$$r \le \frac{d(y_n, y_m)}{|y_n| \vee |y_m| \vee 1}, \frac{d(y'_n, y'_m)}{|y'_n| \vee |y'_m| \vee 1}, \frac{d(y_n, y'_m)}{|y_n| \vee |y'_m| \vee 1}. \tag{8.5}$$

Combining (8.3). (8.4) and (8.5) gives

$$d(b_i^n, b_j^m) \le \frac{K}{r} d(y_n, y_m), \frac{2K}{r} d(y'_n, y'_m), \frac{2K}{r} d(y_n, y'_m) \text{ if } m \neq n.$$

Finally, for any n,

$$d(b_1^n, b_2^n) \le \frac{4d(v_n, \widetilde{v_n})}{k_n} \le 4d(y_n, y'_n).$$

This completes the proof of the claim.

For each n, choose $d^n \in X_1$ so that $\Phi(y'_n, d^n) = b_1^n$. Then

$$\begin{aligned}(y_n, y'_n, b_1^n, b_1^n) &= (\varphi \cdot \Phi)(x_n, x'_n, a_1^n, d^n) \text{ and}\\ (y_n, y'_n, b_2^n, b_1^n) &= (\varphi \cdot \Phi)(x_n, x'_n, a_2^n, d^n).\end{aligned}$$

By the claim and Theorem 7.6, there exists an infinite set $I \subseteq \mathbb{N}$ so that both $(x_n, x'_n, a_1^n, d^n)_{n\in I}$ and $(x_n, x'_n, a_2^n, d^n)_{n\in I}$ are L-sequences. Hence there exists $C_2 < \infty$ so that for any $n \in I$,

$$\frac{c_1|x_n|}{k_n} \le \frac{d(u, \widetilde{u_n})}{k_n} \le d(a_1^n, a_2^n) \le d(a_1^n, d^n) + d(d^n, a_2^n) \le C_2 d(x_n, x'_n).$$

On the other hand, by (8.2)

$$k_n d(y_n, y'_n) \le 2d(v_n, \widetilde{v_n}) \le 2(\mathrm{Lip}(Rg) + \mathrm{Lip}(Rf))|y_n| + 2d(Rg(y_0), Rf(y_0)).$$

Thus there exists $C_3 < \infty$ so that

$$\rho(y_n, y'_n) = \frac{d(y_n, y'_n)}{|y_n| \vee |y'_n| \vee 1} \le \frac{C_3}{k_n} \le \frac{C_2 C_3}{c_1} \cdot \frac{d(x_n, x'_n)}{|x_n|} \text{ for any } n \in I.$$

Since $|x_n| \ge |x'_n| \vee r$ by choice of the subsequence, there is a constant $C_4 < \infty$ so that $\frac{d(x_n, x'_n)}{|x_n|} \le C_4 \rho(x_n, x'_n)$ for all n. Thus $(\frac{\rho(y_n, y'_n)}{\rho(x_n, x'_n)})_{n\in I}$ is bounded. This completes the proof in Case 1.

Case 2. $d(v_n, \widetilde{v_n}) \le d(y_n, y'_n)$ for infinitely many n.

We may assume that $d(v_n, \widetilde{v_n}) \le d(y_n, y'_n)$ for all n. Furthermore, if $\inf \rho(x_n, x'_n) > 0$, the result is obvious since ρ is a bounded metric on X_2. Thus we may assume that $\rho(x_n, x'_n) \to 0$. If $m \neq n$, then

$$\begin{aligned}d(\widetilde{v_n}, Rg(y'_m)) &\le d(Rf(y_n), Rf(y_0)) + d(Rf(y_0), Rg(y_0))\\ &\qquad + d(Rg(y_0), Rg(y'_m))\\ &\le M(|y_n| \vee 1 \vee |y'_m|)\\ &\qquad \text{for some constant } M \text{ independent of } m, n\\ &\le \frac{Md(y_n, y'_m)}{r} \text{ since } \rho(y_n, y'_m) \ge r.\end{aligned}$$

Also,

$$d(\widetilde{v_n}, Rg(y'_n)) \le d(\widetilde{v_n}, v_n) + d(Rg(y_n), Rg(y'_n)) \le Kd(y_n, y'_n)$$

for some constant $K < \infty$ and Rf and Rg are Lipschitz functions. Thus the sequence $(y_n, y'_n, \widetilde{v_n}, Rg(y'_n))_n$ is an L-sequence. By Proposition 7.5, we find $h \in \mathrm{Lip}(X_2, E_2)$ and an infinite set $I \subseteq \mathbb{N}$ so that $h(y_n) = \widetilde{v_n}$ and $h(y'_n) = Rg(y'_n)$ for all $n \in I$. Hence $R^{-1}h(x_n) = \widetilde{u_n}$ and $R^{-1}h(x'_n) = g(x'_n) = u$ for all $n \in I$. Thus

$$c_1|x_n| \le d(\widetilde{u_n}, u) \le \mathrm{Lip}(R^{-1}h)\, d(x_n, x'_n).$$

Recall that $|x_n| \ge |x'_n| \vee r$ for all n, where we may assume that $0 < r \le 1$. Then $|x_n| \ge r(|x_n| \vee |x'_n| \vee 1)$. Hence

$$\rho(x_n, x'_n) = \frac{d(x_n, x'_n)}{|x_n| \vee |x'_n| \vee 1} \ge \frac{rc_1}{\mathrm{Lip}(R^{-1}h)},$$

contradicting the fact that $\rho(x_n, x'_n) \to 0$. □

Lemma 8.7. *Suppose that (y_n) and (y'_n) are sequences in X_2 convergent to y, with $d(y'_n, y) \ge d(y_n, y)$ and $y_n \ne y'_n$ for all n. For any $v \in E_2$, there exists $k = k(v) \in \mathbb{N}$ so that if $n \ge k$, $d(v, v') \le d(y_n, y'_n)$ and $(y_n, y'_n, v, v') = (\varphi \cdot \Phi)(x_n, x'_n, u, u')$, then $d(u, u') \le kd(x_n, x'_n)$.*

Proof. We may assume that either $y_n \ne y$ for all n or that $y_n = y$ for all n. By taking a subsequence, we may also assume that $2d(y'_{n+1}, y) \le d(y'_n, y)$; even $2d(y'_{n+1}, y) \le d(y_n, y)$ if $y_n \ne y$.

If the lemma fails, we can find $v \in E_2$, $n_1 < n_2 < \cdots$ and $(v'_k) \subseteq E_2$ so that $d(v, v'_k) \le d(y_{n_k}, y'_{n_k})$ and $d(u_k, u'_k) > kd(x_{n_k}, x_{n'_k})$ for all k, where $(y_{n_k}, y'_{n_k}, v, v'_k) = (\varphi \cdot \Phi)(x_{n_k}, x'_{n_k}, u_k, u'_k)$.

Claim. $(y_{n_k}, y'_{n_k}, v, v'_k)_k$ is an L-sequence.

It is required to find a constant $K < \infty$ so that for any j, k,

(1) $d(v, v) \le Kd(y_{n_j}, y_{n_k})$,
(2) $d(v'_j, v'_k) \le Kd(y'_{n_j}, y'_{n_k})$, and
(3) $d(v, v'_k) \le Kd(y_{n_j}, y'_{n_k})$.

Item (1) is trivial. For item (2), we may assume that $j < k$. Then

$$\begin{aligned}d(v'_j, v'_k) &\le d(v'_j, v) + d(v'_k, v) \le d(y_{n_j}, y'_{n_j}) + d(y_{n_k}, y'_{n_k})\\ &\le d(y_{n_j}, y) + d(y'_{n_j}, y) + d(y_{n_k}, y) + d(y'_{n_k}, y)\\ &\le 2d(y'_{n_j}, y) + 2d(y'_{n_k}, y) \le 4d(y'_{n_j}, y).\end{aligned}$$

On the other hand,

$$d(y'_{n_j}, y'_{n_k}) \geq d(y'_{n_j}, y) - d(y'_{n_k}, y) \geq \frac{1}{2} d(y'_{n_j}, y).$$

Thus (2) holds with $K = 8$.

For item (3), if $y_n = y$ for all n, then

$$d(v, v'_k) \leq d(y_{n_k}, y'_{n_k}) = d(y, y'_{n_k}) = d(y_{n_j}, y'_{n_k}).$$

Suppose that $y_n \neq y$ for all n. If $j = k$, then (3) holds with $K = 1$. Assume that $j \neq k$. We have

$$\begin{aligned} d(y_{n_j}, y'_{n_k}) \geq |d(y_{n_j}, y) - d(y'_{n_k}, y)| &\geq \begin{cases} \frac{1}{2} d(y_{n_j}, y) & \text{if } j < k \\ \frac{1}{2} d(y'_{n_k}, y) & \text{if } j > k \end{cases} \\ &\geq \frac{1}{4} d(y'_{n_k}, y). \end{aligned}$$

Since

$$d(v, v'_k) \leq d(y_{n_k}, y'_{n_k}) \leq d(y_{n_k}, y) + d(y'_{n_k}, y) \leq 2d(y'_{n_k}, y),$$

item (3) holds with $K = 8$. This completes the proof of the claim.

By the claim and Theorem 7.6, $(x_{n_k}, x'_{n_k}, u_k, u'_k)_k$ has an L-subsequence. This contradicts the choice that $d(u_k, u'_k) > kd(x_{n_k}, x_{n'_k})$ for all k. $\square$

Proposition 8.8. *Let (x_n), (x'_n) be sequences in X_1 and let $y_n = \varphi(x_n)$, $y'_n = \varphi(x'_n)$. Assume that (x_n) and (x'_n) are both convergent to the same $x \in X_1$ and that $x_n \neq x'_n$ for all n. Then $(\frac{\rho(y_n, y'_n)}{\rho(x_n, x'_n)})$ is bounded.*

Proof. Since φ is continuous, (y_n) and (y'_n) both converge to $y = \varphi(x)$. We may assume that $d(y'_n, y) \geq d(y_n, y)$ for all n. For each $v \in E_2$, let $k(v)$ be obtained from Lemma 8.7. Let F_k be the set of all $v \in E_2$ such that $k(v) = k$. By Lemma 8.7, $\bigcup_k F_k = E_2$. By the Baire Category Theorem, there exists $k_0 \in \mathbb{N}$ so that $\overline{F_{k_0}}$ has nonempty interior.

Choose $v \in E_2$ and $r > 0$ so that $B(v, 2r) \subseteq \overline{F_{k_0}}$. By path connectedness, there exists $\widetilde{v} \in B(v, r)$, $\widetilde{v} \neq v$. Since $d(y_n, y'_n) \to 0$, we may assume that $d(y_n, y'_n) < 2d(v, \widetilde{v})$ for all n. For each n, let $k_n \in \mathbb{N}$ be such that $2d(v, \widetilde{v}) < k_n d(y_n, y'_n) \leq 4d(v, \widetilde{v})$. Determine $u_n, \widetilde{u_n}$ by $(\varphi \cdot \Phi)(x'_n, x'_n, u_n, \widetilde{u_n}) = (y'_n, y'_n, v, \widetilde{v})$. Apply Lemma 8.5 to find $a_1^n, a_2^n \in E_1$ so that

$$d(a_1^n, a_2^n) \geq \frac{d(u_n, \widetilde{u_n})}{k_n}, \; d(b_1^n, b_2^n) \leq \frac{2d(v, \widetilde{v})}{k_n} \text{ and } d(b_i^n, v) \leq 2d(v, \widetilde{v}),$$

where $b_i^n = \Phi(y'_n, a_i^n)$. Note that

$$d(b_1^n, b_2^n) < d(y_n, y'_n) \text{ and } b_i^n \in B(v, 2r) \subseteq \overline{F_{k_0}}.$$

Hence there exists $e^n \in F_{k_0}$ so that $d(e^n, b_i^n) < d(y_n, y'_n)$, $i = 1, 2$. Let c^n be such that $\Phi(y_n, c^n) = e^n$. Since $k(e^n) = k_0$ and $d(e^n, b_i^n) < d(y_n, y'_n)$, $i = 1, 2$, $d(c^n, a_i^n) \leq k_0 d(x_n, x'_n)$ if $n \geq k_0$. Therefore,

$$\frac{d(u_n, \widetilde{u_n})}{k_n} \leq d(a_1^n, a_2^n) \leq 2k_0 d(x'_n, x_n).$$

However, $d(y_n, y'_n) \leq \frac{4d(v, \widetilde{v})}{k_n}$. Thus

$$d(y_n, y'_n) \leq \frac{8k_0\, d(v, \widetilde{v})}{d(u_n, \widetilde{u_n})}\, d(x_n, x'_n) < \frac{8k_0 r\, d(x_n, x'_n)}{d(u_n, \widetilde{u_n})} \quad \text{if } n \geq k_0.$$

Consider the constant functions with values v and $\widetilde{v}$ on X_2, which we denote by the same symbols v and $\widetilde{v}$. Then $u_n = R^{-1}v(x_n)$ and $\widetilde{u_n} = R^{-1}\widetilde{v}(x'_n)$. Hence (u_n) and $(\widetilde{u_n})$ converge to $u := R^{-1}v(x)$ and $\widetilde{u} := R^{-1}\widetilde{v}(x)$ respectively. Since $\Phi(y, u) = v \neq \widetilde{v} = \Phi(y, \widetilde{u})$ and $\Phi(y, \cdot)$ is a bijection, $d(u_n, \widetilde{u_n}) \to d(u, \widetilde{u}) \neq 0$. It follows that $(\frac{d(y_n, y'_n)}{d(x_n, x'_n)})$ is bounded. Finally, $|y_n| \vee |y'_n| \vee 1 \to |y| \vee 1$ and $|x_n| \vee |x'_n| \vee 1 \to |x| \vee 1$. Thus we see that $(\frac{\rho(y_n, y'_n)}{\rho(x_n, x'_n)})$ is bounded. □

Proof of Theorem 8.2. It suffices to show that $\varphi : (X_1, \rho) \to (X_2, \rho)$ is Lipschitz. The result then follows by symmetry. Suppose that φ fails to be ρ-to-ρ Lipschitz. Then there are sequences $(x_n), (x'_n)$ in X_1 so that $\rho(y_n, y'_n) > n\rho(x_n, x'_n)$ for all n, where $y_n = \varphi(x_n)$ and $y'_n = \varphi(x'_n)$. Since ρ is a bounded function on X_2, $\rho(x_n, x'_n) \to 0$. As ρ is within a multiple of a metric topologically equivalent to d, convergence with respect to ρ and d are the same. If one of the sequences (y_n) or (y'_n) has a convergent subsequence, then (x_n) or (x'_n) has a convergent subsequence, since φ is a homeomorphism. As $\rho(x_n, x'_n) \to 0$, there exists an infinite set $I \subseteq \mathbb{N}$ so that $(x_n)_{n\in I}$ and $(x'_n)_{n\in I}$ are both convergent to the same $x \in X_1$. In this case, $(\frac{\rho(y_n, y'_n)}{\rho(x_n, x'_n)})_{n\in I}$ is bounded by Proposition 8.8, a contradiction to the choices of (x_n) and (x'_n). On the other hand, if neither (y_n) nor (y'_n) have convergent subsequences, then we may assume that both (y_n) and (y'_n) are ρ-separated sequences. A combination of Propositions 8.4 and 8.6 show that $(\frac{\rho(y_n, y'_n)}{\rho(x_n, x'_n)})$ has a bounded subsequence, reaching yet another contradiction. □

8.2. Continuity of Φ

Let X_i, E_i, $i = 1, 2$, R, φ and Φ be as in the previous subsection. It is known that the map Φ need not be continuous. In fact, [1, p. 190] gives an example of a $\perp$-isomorphism $R : \mathrm{Lip}(X, \mathbb{R}) \to \mathrm{Lip}(X, \mathbb{R})$, where $X = [0, 1]$ with the Hölder metric $d(x, y) = |x - y|^\alpha$, $0 < \alpha < 1$, so that the associated map $\Phi(y, \cdot) : \mathbb{R} \to \mathbb{R}$ is not continuous for some $y \in X$. Nevertheless, we will show that Φ is continuous at almost all points in the sense of Baire category.

Lemma 8.9. *Let $(x, u) \in X_1 \times E_1$. There exists $k = k(x, u) \in \mathbb{N}$ so that*

$$d(u', u) \leq d(x', x) \leq \frac{1}{k} \implies d(\Phi(\varphi(x'), u'), \Phi(\varphi(x), u)) \leq kd(\varphi(x'), \varphi(x)).$$

Proof. If the lemma fails, we can find a sequence $(x'_k, u'_k)_k$ in $X_1 \times E_1$ so that $d(u'_k, u) \leq d(x'_k, x) \leq \frac{1}{k}$, but $d(v'_k, v) > kd(y'_k, y)$, where $y'_k = \varphi(x'_k)$, $y = \varphi(x)$, $v'_k = \Phi(y'_k, u'_k)$ and $v = \Phi(y, u)$. Note that $x'_k \neq x$ for all k. Apply Lemma 8.7 with the sequence (x'_k) and the constant sequence $(x_k) = (x)$ in X_1 instead of X_2. Since $d(u'_k, u) \leq d(x'_k, x)$ for all k, we obtain $k_0 \in \mathbb{N}$ so that if $k \geq k_0$, then $d(v'_k, v') \leq k_0\, d(y'_k, y)$. This contradicts the choice of $(x'_k, u'_k)_k$ and completes the proof of the lemma. □

Theorem 8.10. *Let $R \subseteq \mathrm{Lip}(X_1, E_1) \times \mathrm{Lip}(X_2, E_2)$ be a $\perp$-isomorphism so that $\pi_i(R) = \mathrm{Lip}(X_i, E_i)$, $i = 1, 2$, where X_i, E_i are complete metric spaces. Suppose that Φ is the associated map given by Theorem 5.2. There is a dense open set O in $X_1 \times E_1$ so that the map $(x, u) \mapsto \Phi(\varphi(x), u) : X_1 \times E_1 \to E_2$, is continuous at (x, u) if x is an accumulation point of X_1 and $(x, u) \in O$.*

Proof. For any $(x, u) \in E_1$, let $F_k = \{(x, u) \in X_1 \times E_1 : k(x, u) = k\}$. By Lemma 8.9, $\bigcup F_k = X_1 \times E_1$. Hence it follows from the Baire Category Theorem that $O := \bigcup \mathrm{int}\, \overline{F_k}$ is a dense open set in $X_1 \times E_1$. Assume that $(x, u) \in O$ and that x is an accumulation point. To prove the theorem, it suffices to show that if $(x_n, u_n) \to (x, u)$ in $X_1 \times E_1$, then $(\Phi(\varphi(x_n), u_n))$ converges to $\Phi(\varphi(x), u)$. Choose k_0 so that $(x, u) \in \mathrm{int}\, \overline{F_{k_0}}$. Fix $r > 0$ so that $(x, u') \in \overline{F_{k_0}}$ if $d(u', u) \leq r$. For each n, let u_n denote the constant function with value

u_n on E_1. Then $\Phi(\varphi(z), u_n) = Ru_n(\varphi(z))$ is a continuous function of z. If $x_n = x$, then since x is an accumulation point, there exists z_n so that $0 < d(z_n, x_n) \to 0$ and $d(\Phi(\varphi(z_n), u_n), \Phi(\varphi(x_n), u_n)) \to 0$. Thus, without loss of generality, we may assume additionally that $x_n \neq x$ for any n. Let $\varepsilon > 0$ be given. There exists z such that $0 < d(z, x) \leq \frac{1}{k_0}$ and that

$$d(\varphi(z), \varphi(x)),\ d(\Phi(\varphi(z), u), \Phi(\varphi(x), u)) < \varepsilon.$$

Choose $N \in \mathbb{N}$ so that $d(u_n, u) < d(z, x) \wedge r$ and $d(x_n, x) \leq \frac{1}{k_0}$ if $n \geq N$. If $n \geq N$, then $(x, u_n) \in \overline{F_{k_0}}$ and $d(u_n, u) < d(z, x)$. Hence there exists $(x, \widetilde{u_n}) \in F_{k_0}$ so that $d(u, \widetilde{u_n}) \leq d(z, x) \leq \frac{1}{k_0}$ and $d(u_n, \widetilde{u_n}) \leq d(x_n, x) \leq \frac{1}{k_0}$. By definition of F_{k_0},

$$d(\Phi(\varphi(z), u), \Phi(\varphi(x), \widetilde{u_n})) \leq k_0 d(\varphi(z), \varphi(x)) \leq k_0\varepsilon$$
$$d(\Phi(\varphi(x_n), u_n), \Phi(\varphi(x), \widetilde{u_n}))) \leq k_0 d(\varphi(x_n), \varphi(x)).$$

Therefore,

$$d(\Phi(\varphi(x_n), u_n), \Phi(\varphi(x), u)) \leq \varepsilon + k_0\varepsilon + k_0 d(\varphi(x_n), \varphi(x)).$$

As $\varphi(x_n) \to \varphi(x)$ and ε is arbitrary, $\Phi(\varphi(x_n), u_n) \to \Phi(\varphi(x), u)$. $\square$

Final remark and acknowledgment.

In the latter parts of the paper, we consider arc length spaces with uniform/Lipschitz local contractibility. If E' is a complete metric space that is uniformly homeomorphic to an arc length space E that is uniformly locally contractible, then the spaces $U(X, E')$ and $U(X, E)$ are naturally $\perp$-isomorphic. So all results in the paper pertaining to $U(X, E)$ also apply to $U(X, E')$. A similar remark holds for the Lipschitz case.

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