Banach-Stone Theorems for disjointness preserving relations

Denny H. Leung and Wee Kee Tang

Abstract. The concept of disjointness preserving mappings has proved to be a useful unifying idea in the study of Banach-Stone type theorems. In this paper, we examine disjointness preserving relations between sets of continuous functions (valued in general topological spaces). Under very mild assumptions, it is shown that a disjointness preserving relation is completely determined by a Boolean isomorphism between the Boolean algebras of regular open sets in the domain spaces. Building on this result, certain Banach-Stone type theorems are obtained for disjointness preserving relations. From these, we deduce a generalization of Kaplansky's classical theorem concerning order isomorphisms to sets of continuous functions with values topological lattices. As another application, we prove some results on the characterization of nonvanishing preservers. Throughout, the domains of the function spaces need not be compact.

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1. Introduction

If Ω is a compact Hausdorff space, the space $C(\Omega)$ of continuous real valued functions on Ω contains rich information on its underlying space Ω. In particular, $C(\Omega)$ carries with it a wealth of structures. It is a Banach space under the norm $\|f\| = \sup_{\omega\in\Omega} |f(\omega)|$, a ring (with unit) under pointwise addition and multiplication and a vector lattice under pointwise order. Each of these aspects of $C(\Omega)$ has been shown to determine the space Ω up to homeomorphism. These are the famous classical theorems of Banach-Stone

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[6, 29], Gelfand-Kolmogorov [15] and Kaplansky [21]. These three classical theorems can be summarized as follows.

Theorem 1.1. *Let Ω, Σ be compact Hausdorff spaces and let $T : C(\Omega) \to C(\Sigma)$ be a bijection. If T is a*

(1) *linear isometry (Banach-Stone), or*
(2) *algebra isomorphism (Gelfand-Kolmogorov), or*
(3) *vector lattice isomorphism (Kaplansky),*

then there are a homeomorphism $\varphi : \Sigma \to \Omega$ and a function $h \in C(\Sigma)$ so that

$$Tf(\sigma) = h(\sigma)f(\varphi(\sigma)) \text{ for all } f \in C(\Omega) \text{ and all } \sigma \in \Sigma.$$

Moreover $|h| = 1$, $h = 1$ and $h > 0$ under assumptions (a), (b), (c) respectively.

A particularly fruitful concept that unifies the three classical theorems is that of disjointness preserving operators. Two functions $f, g \in C(\Omega)$ are said to be *disjoint* if the pointwise product $fg = 0$. Suppose that $A(\Omega)$ and $A(\Sigma)$ are vector subspaces of $C(\Omega)$ and $C(\Sigma)$ respectively. A linear operator $T : A(\Omega) \to A(\Sigma)$ is *disjointness preserving* if Tf, Tg are disjoint whenever f, g are disjoint functions in $A(\Omega)$. A *biseparating* operator or $\perp$-*isomorphism* (see [10]) is a linear bijection $T : A(\Omega) \to A(\Sigma)$ so that both T and T^{-1} are disjointness preserving. It is evident that if $A(\Omega)$ and $A(\Sigma)$ are algebras under pointwise operations, then every algebraic isomorphism $T : A(\Omega) \to A(\Sigma)$ is biseparating. A similar statement holds for lattice isomorphisms. In the past three decades, many results concerning biseparating maps have been obtained. See, e.g., [1, 2, 3, 4, 5, 14, 18, 19, 20]. Most of these are in the context of linear or additive maps. Recently, the papers [10, 13] appeared that took the study of isomorphisms of disjointness structure to very general settings. For a recent survey on disjointness preservers and Banach-Stone Theorems, see [24]. In particular, the paper [10] shows how disjointness preserving mappings can be used to prove and/or extend many results regarding "preservers" of various sorts. The purpose of this paper is to build and expand on the ideas found in [10, 13]. We obtain several results of Banach-Stone type for relations that are $\perp$-isomorphims. Applications to order isomorphisms, generalizing Kaplansky's classical theorem, as well as to nonvanishing preservers are given.

In §2, we extend the definition of $\perp$-isomorphism to a general relation R between sets of continuous functions. The first main result (Theorem 2.4) shows that as long as there are sufficiently many elements in R, a $\perp$-isomorphism is characterized by a Boolean isomorphism between the Boolean algebras of regular open sets in the domains of definition.

§3 examines the question of when a Boolean isomorphism between the Boolean algebras of regular open sets in the topological spaces X_1 and X_2 respectively gives rise to a homeomorphism $\varphi : X_1 \to X_2$. Combining the results in §2 and §3 leads to several results of Banach-Stone type. See Theorems 4.2, 4.3 and 4.5 in §4.

The paper ends with two applications. In §5, it is shown that under general assumptions, an order isomorphism between lattices of continuous functions taking values in topological lattices are $\perp$-isomorphisms. As a result we obtain in Theorem 5.4 a generalization of Kaplansky's classical result to topological lattice-valued lattices of continuous functions. §6 contains the application to nonvanishing preservers, which has been studied in, for instance, [10, 11, 17, 22, 25], sometimes under the name of maps preserving common zeros. Proposition 6.1 shows that a nonvanishing preservers is a $\perp$-isomorphism in many instances. As a result, we obtain Banach-Stone type theorems for nonvanishing preservers (Theorems 6.5 and 6.7).

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2. Characterization of $\perp$-isomorphisms in terms of regular open sets

A topological space is **Tychonoff** if it is Hausdorff and completely regular. In this section, let X_1, X_2 be Tychonoff spaces and E_1, E_2 be Hausdorff spaces. If $f, g, h \in C(X, E)$, the set of continuous E-valued functions on X, let $[f \neq h] = \{x \in X : f(x) \neq h(x)\}$ and $\sigma_h(f) = \operatorname{int} \overline{[f \neq h]}$. Say that

1. $f \perp_h g$ if $[f \neq h] \cap [g \neq h] = \emptyset$.
2. $f \subseteq_h g$ if $\sigma_h(f) \subseteq \sigma_h(g)$.

Note that $f \perp_h g$ if and only if $\sigma_h(f) \cap \sigma_h(g) = \emptyset$. Let $R \subseteq C(X_1, E_1) \times C(X_2, E_2)$ be a relation and suppose that $(h_1, h_2) \in R$. R is a $\perp_{h_1,h_2}$-**isomorphism**, respectively, a $\subseteq_{h_1,h_2}$**-isomorphism**, if for any $(f_1, f_2), (g_1, g_2) \in R$, $f_1 \perp_{h_1} g_1 \iff f_2 \perp_{h_2} g_2$, respectively, $f_1 \subseteq_{h_1} g_1 \iff f_2 \subseteq_{h_2} g_2$. Denote the projection from $C(X_1, E_1) \times C(X_2, E_2)$ onto $C(X_i, E_i)$ by π_i. Set $A(X_i, E_i) = \pi_i(R)$, $i = 1, 2$ and let $\Sigma_{h_i} = \{\sigma_{h_i}(f_i) : f_i \in A(X_i, E_i)\}$. Say that $A(X_i, E_i)$ is h_i**-weakly regular** if Σ_{h_i} is a basis for the topology of X_i. R is (h_1, h_2)**-weakly regular** if $A(X_i, E_i)$ is h_i-weakly regular for $i = 1, 2$.

The following simple yet fundamental result was used in [10]; its ancestry goes back to at least [2].

Proposition 2.1. *Suppose that R is (h_1, h_2)-weakly regular. Then R is a $\perp_{h_1,h_2}$-isomorphism if and only if it is a $\subseteq_{h_1,h_2}$-isomorphism.*

Proof. Suppose that R is a $\perp_{h_1,h_2}$-isomorphism. Assume that there are (f_1, f_2), $(g_1, g_2) \in R$ so that $\sigma_{h_1}(f_1) \subseteq \sigma_{h_1}(g_1)$ but $\sigma_{h_2}(f_2) \not\subseteq \sigma_{h_2}(g_2)$, so that $\sigma_{h_2}(f_2) \not\subseteq \overline{[g_2 \neq h_2]}$. Pick y and $k_2 \in A(X_2, E_2)$ so that $y \in \sigma_{h_2}(k_2) \subseteq \sigma_{h_2}(f_2) \backslash \overline{[g_2 \neq h_2]}$. Let k_1 be such that $(k_1, k_2) \in R$. Since $k_2 \perp_{h_2} g_2$, $k_1 \perp_{h_1} g_1$. Hence $k_1 \perp_{h_1} f_1$, which implies that $k_2 \perp_{h_2} f_2$. This contradicts the fact that $y \in \sigma_{h_2}(f_2) \cap \sigma_{h_2}(k_2)$. This shows that $\sigma_{h_1}(f_1) \subseteq \sigma_{h_1}(g_1) \implies \sigma_{h_2}(f_2) \subseteq \sigma_{h_2}(g_2)$. By symmetry, R is a $\subseteq_{h_1,h_2}$-isomorphism. Conversely, suppose that R is a $\subseteq_{h_1,h_2}$-isomorphism, but there are (f_1, f_2), $(g_1, g_2) \in R$ so that $f_1 \perp_{h_1} g_1$ but $\sigma_{h_2}(f_2) \cap \sigma_{h_2}(g_2) \neq \emptyset$. There exists $(k_1, k_2) \in R$ so that $\emptyset \neq \sigma_{h_2}(k_2) \subseteq \sigma_{h_2}(f_2) \cap \sigma_{h_2}(g_2)$. Thus $\sigma_{h_1}(k_1) \subseteq \sigma_{h_1}(f_1) \cap \sigma_{h_1}(g_1)$.

Since $f_1 \perp_{h_1} g_1$, $\sigma_{h_1}(k_1) = \emptyset$. In particular, $\sigma_{h_1}(k_1) \subseteq \sigma_{h_1}(h_1)$ and thus $\sigma_{h_2}(k_2) \subseteq \sigma_{h_2}(h_2) = \emptyset$, contrary to the choice of k_2. This shows that $f_1 \perp_{h_1} g_1 \implies f_2 \perp_{h_2} g_2$. By symmetry, R is a $\perp_{h_1,h_2}$-isomorphism. $\square$

In a Tychonoff space X, an open set $U \subseteq X$ is a **regular open set** if $U = \operatorname{int} \overline{U}$. Note that all sets of the form $\sigma_h(f)$ are regular open sets. The collection $\mathrm{RO}(X)$ of all regular open sets in X is a Boolean algebra with the operations $U_1 \wedge U_2 = U_1 \cap U_2$, $U_1 \vee U_2 = \operatorname{int} \overline{U_1 \cup U_2}$ and $\neg U = \operatorname{int}(U^c)$. Moreover, $\mathrm{RO}(X)$ is **complete**; that is, every nonempty subset of $\mathrm{RO}(X)$ has a supremum. Refer to [28] for an exposition of the theory of Boolean algebras. The next result shows that every $\perp_{h_1,h_2}$-isomorphism determines a Boolean isomorphism $\theta_{h_1,h_2} : \mathrm{RO}(X_1) \to \mathrm{RO}(X_2)$.

Proposition 2.2. *Let R be a (h_1, h_2)-weakly regular $\perp_{h_1,h_2}$-isomorphism. Define $\theta_{h_1,h_2} : \mathrm{RO}(X_1) \to \mathrm{RO}(X_2)$ by*

$$\theta_{h_1,h_2}(U) = \operatorname{int} \overline{\bigcup \{\sigma_{h_2}(f_2) : (f_1, f_2) \in R, \sigma_{h_1}(f_1) \subseteq U\}}.$$

Then θ_{h_1,h_2} is a Boolean isomorphism from $\mathrm{RO}(X_1)$ onto $\mathrm{RO}(X_2)$ so that for any $(f_1, f_2) \in R$ and any $U \in \mathrm{RO}(X_1)$, $f_1 = h_1$ on U if and only if $f_2 = h_2$ on $\theta_{h_1,h_2}(U)$.

Proof. Clearly θ_{h_1,h_2} is order preserving in the sense that if $U_1, U_2 \in \mathrm{RO}(X_1)$ and $U_1 \subseteq U_2$, then $\theta_{h_1,h_2}(U_1) \subseteq \theta_{h_1,h_2}(U_2)$. Define $\omega : \mathrm{RO}(X_2) \to \mathrm{RO}(X_1)$ by

$$\omega(V) = \operatorname{int} \overline{\bigcup \{\sigma_{h_1}(f_1) : (f_1, f_2) \in R, \sigma_{h_2}(f_2) \subseteq V\}}.$$

We claim that $\omega(\theta_{h_1,h_2}(U)) = U$ for any $U \in \mathrm{RO}(X_1)$. By symmetry, we would also have $\theta_{h_1,h_2}(\omega(V)) = V$ for any $V \in \mathrm{RO}(X_2)$. Hence θ_{h_1,h_2} is a Boolean isomorphism. Let us prove the claim. Assume that $x \in U \in \mathrm{RO}(X_1)$. By weak regularity, there exists $(f_1, f_2) \in R$ so that $x \in \sigma_{h_1}(f_1) \subseteq U$. In particular, $\sigma_{h_2}(f_2) \subseteq \theta_{h_1,h_2}(U)$. Thus $x \in \sigma_{h_1}(f_1) \subseteq \omega(\theta_{h_1,h_2}(U))$. This proves that $U \subseteq \omega(\theta_{h_1,h_2}(U))$. Conversely, suppose that $\omega(\theta_{h_1,h_2}(U)) \backslash \overline{U} \neq \emptyset$. By weak regularity, there exists $(f_1, f_2) \in R$ so that $\emptyset \neq \sigma_{h_1}(f_1) \subseteq \omega(\theta_{h_1,h_2}(U)) \backslash \overline{U}$. By definition of ω, there exists $(g_1, g_2) \in R$ so that $\sigma_{h_2}(g_2) \subseteq \theta_{h_1,h_2}(U)$ and that $\sigma_{h_1}(f_1) \cap \sigma_{h_1}(g_1) \neq \emptyset$. Thus $\sigma_{h_2}(f_2) \cap \sigma_{h_2}(g_2) \neq \emptyset$. By weak regularity again, choose $(k_1, k_2) \in R$ so that $\emptyset \neq \sigma_{h_2}(k_2) \subseteq \sigma_{h_2}(f_2) \cap \sigma_{h_2}(g_2)$. By Proposition 2.1, $\sigma_{h_1}(k_1) \subseteq \sigma_{h_1}(f_1) \cap \sigma_{h_1}(g_1)$. In particular, $\sigma_{h_1}(k_1) \cap U \subseteq \sigma_{h_1}(f_1) \cap U = \emptyset$. However, $\sigma_{h_2}(k_2) \subseteq \sigma_{h_2}(g_2) \subseteq \theta_{h_1,h_2}(U)$. Hence there exists $(l_1, l_2) \in R$ so that $\sigma_{h_1}(l_1) \subseteq U$ and that $\sigma_{h_2}(k_2) \cap \sigma_{h_2}(l_2) \neq \emptyset$. This implies that $\sigma_{h_1}(k_1) \cap \sigma_{h_1}(l_1) \neq \emptyset$ and hence $\sigma_{h_1}(k_1) \cap U \neq \emptyset$, contrary to what was established above. We have shown that $\omega(\theta_{h_1,h_2}(U)) \subseteq \overline{U}$. Since $U \in \mathrm{RO}(X)$, $\omega(\theta_{h_1,h_2}(U)) \subseteq U$. The proof of the claim is complete. Thus θ_{h_1,h_2} is a Boolean isomorphism. Finally, assume that $(f_1, f_2) \in R$ and $f_1 = h_1$ on $U \in \mathrm{RO}(X_1)$. Then $\sigma_{h_1}(f_1) \subseteq \neg U = \operatorname{int}(U^c)$. By definition of θ_{h_1,h_2}, $\sigma_{h_2}(f_2) \subseteq \theta_{h_1,h_2}(\neg U) = \neg \theta_{h_1,h_2}(U) = \operatorname{int}(\theta_{h_1,h_2}(U)^c)$. Therefore, $f_2 = h_2$ on $\theta_{h_1,h_2}(U)$. The converse follows by symmetry. $\square$

In terms of the order on $\mathrm{RO}(X_2)$, $\theta_{h_1,h_2}(U)$ can be described by

$$\theta_{h_1,h_2}(U) = \bigvee\{\sigma_{h_2}(f_2) : (f_1, f_2) \in R, \sigma_{h_1}(f_1) \subseteq U\}.$$

In fact, existence of a Boolean isomorphism satisfying the last condition of Proposition 2.2 implies that R is a $\perp_{h_1,h_2}$-isomorphism. In order to see this, we first prove an easy lemma.

Lemma 2.3. *Let R be (h_1, h_2)-weakly regular. Suppose that $\theta : \mathrm{RO}(X_1) \to \mathrm{RO}(X_2)$ is a Boolean isomorphism so that for any $(f_1, f_2) \in R$ and any $U \in \mathrm{RO}(X_1)$, $f_1 = h_1$ on $U \iff f_2 = h_2$ on $\theta(U)$. Then for any $(g_1, g_2) \in R$, $\sigma_{h_2}(g_2) = \theta(\sigma_{h_1}(g_1))$.*

Proof. It suffices to show that $\sigma_{h_2}(g_2) \subseteq \overline{\theta(\sigma_{h_1}(g_1))}$. The reverse inclusion follows by symmetry. Assume that $y \notin \overline{\theta(\sigma_{h_1}(g_1))}$. Choose $(f_1, f_2) \in R$ so that

$$y \in \sigma_{h_2}(f_2) \subseteq [\overline{\theta(\sigma_{h_1}(g_1))}]^c = \neg\theta(\sigma_{h_1}(g_1)).$$

Since $\sigma_{h_2}(f_2) \cap \theta(\sigma_{h_1}(g_1)) = \emptyset$, $\theta^{-1}(\sigma_{h_2}(f_2)) \cap \sigma_{h_1}(g_1) = \emptyset$. Thus $g_1 = h_1$ on $\theta^{-1}(\sigma_{h_2}(f_2))$, which implies that $g_2 = h_2$ on $\sigma_{h_2}(f_2)$ by assumption. In particular, $g_2(y) = h_2(y)$. This shows that $g_2 = h_2$ on the set $[\overline{\theta(\sigma_{h_1}(g_1))}]^c$. Hence $\sigma_{h_2}(g_2) \subseteq \overline{\theta(\sigma_{h_1}(g_1))}$, as required. □

We can now prove the main result of this section characterizing a $\perp_{h_1,h_2}$-isomorphism in terms of an associated Boolean isomorphism $\theta : \mathrm{RO}(X_1) \to \mathrm{RO}(X_2)$.

Theorem 2.4. *Let R be (h_1, h_2)-weakly regular. Then R is a $\perp_{h_1,h_2}$-isomorphism if and only if there is a Boolean isomorphism $\theta : \mathrm{RO}(X_1) \to \mathrm{RO}(X_2)$ such that for any $(f_1, f_2) \in R$ and for any $U \in \mathrm{RO}\,(X_1)$, $f_1 = h_1$ on U if and only if $f_2 = h_2$ on $\theta(U)$.*

Proof. If R is a $\perp_{h_1,h_2}$-isomorphism, then we may take $\theta = \theta_{h_1,h_2}$. The required conclusion follows from Proposition 2.2. Conversely, suppose that there is a Boolean isomorphism $\theta : \mathrm{RO}(X_1) \to \mathrm{RO}(X_2)$ with the given property. Let $(f_1, f_2), (g_1, g_2) \in R$ be such that $f_1 \perp_{h_1} g_1$. Then $f_1 = h_1$ on $\sigma_{h_1}(g_1)$. By assumption, $f_2 = h_2$ on $\theta(\sigma_{h_1}(g_1)) = \sigma_{h_2}(g_2)$, where the last equality follows from Lemma 2.3. Therefore, $f_2 \perp_{h_2} g_2$. By symmetry, we also have $f_2 \perp_{h_2} g_2 \implies f_1 \perp_{h_1} g_1$. □

A Boolean isomorphism satisfying the condition in Theorem 2.4 is said to be **associated with** (R, h_1, h_2), or simply with R. If a Boolean isomorphism $\theta : \mathrm{RO}(X_1) \to \mathrm{RO}(X_2)$ is associated with a $\perp_{h_1,h_2}$-isomorphism R and R is (h_1, h_2)-weakly regular, then it follows easily from Proposition 2.2 and Lemma 2.3 that $\theta = \theta_{h_1,h_2}$.

2.1. Comparing associated Boolean isomorphisms

Suppose that another pair $(k_1, k_2) \in R$ is given, where R is a $\perp_{k_1,k_2}$-isomorphism and $A(X_i, E_i)$ is k_i-weakly regular, $i = 1, 2$. In general, θ_{k_1,k_2} and θ_{h_1,h_2} may be unrelated. Below, we give a sufficient condition in terms of R so that $\theta_{k_1,k_2}(U) = \theta_{h_1,h_2}(U)$ for any $U \in \mathrm{RO}(X_1)$. This will be useful in §4.

Lemma 2.5. *Suppose that R is a (h_1, h_2)- and (k_1, k_2)-weakly regular $\perp_{h_1,h_2}$- and $\perp_{k_1,k_2}$-isomorphism. If $h_1 = k_1$ on a set $U \in \mathrm{RO}(X_1)$, then $\theta_{h_1,h_2}(U) = \theta_{k_1,k_2}(U)$.*

Proof. Let $(f_1, f_2) \in R$ be such that $\sigma_{k_1}(f_1) \subseteq U$. Since

$$[f_1 \neq h_1]\backslash\overline{\sigma_{h_1}(k_1)} \subseteq [f_1 \neq h_1] \cap [h_1 = k_1] \subseteq [f_1 \neq k_1] \subseteq \sigma_{k_1}(f_1),$$

$\sigma_{h_1}(f_1)\backslash\overline{\sigma_{h_1}(k_1)} \subseteq \sigma_{k_1}(f_1)$. In terms of the lattice operations on $\mathrm{RO}(X_1)$, we have $\sigma_{h_1}(f_1) \wedge \neg\sigma_{h_1}(k_1) \subseteq \sigma_{k_1}(f_1)$. Apply θ_{h_1,h_2} to this inclusion and use Lemma 2.3 to obtain

$$\sigma_{h_2}(f_2) \wedge \neg\sigma_{h_2}(k_2) = \theta_{h_1,h_2}(\sigma_{h_1}(f_1)) \wedge \neg\theta_{h_1,h_2}(\sigma_{h_1}(k_1)) \subseteq \theta_{h_1,h_2}(\sigma_{k_1}(f_1)). \tag{2.1}$$

Since $\sigma_{k_1}(f_1) \subseteq U \subseteq [h_1 = k_1]$, $f_1 = k_1$ on $\sigma_{k_1}(h_1)$. Hence $f_2 = k_2$ on $\theta_{k_1,k_2}(\sigma_{k_1}(h_1)) = \sigma_{k_2}(h_2) = \sigma_{h_2}(k_2)$. Thus $\sigma_{k_2}(f_2) \subseteq \overline{\sigma_{h_2}(k_2)}^c$. In particular, if $f_2(y) \neq k_2(y)$, then $k_2(y) = h_2(y)$ and hence $f_2(y) \neq h_2(y)$. So $[f_2 \neq k_2] \subseteq [f_2 \neq h_2]\backslash\overline{\sigma_{h_2}(k_2)}$. Thus $\sigma_{k_2}(f_2) \subseteq \sigma_{h_2}(f_2) \wedge \neg\sigma_{h_2}(k_2)$. Combining with (2.1) gives

$$\sigma_{k_2}(f_2) \subseteq \theta_{h_1,h_2}(\sigma_{k_1}(f_1)) \subseteq \theta_{h_1,h_2}(U).$$

As this holds for any $(f_1, f_2) \in R$ with $\sigma_{k_1}(f_1) \subseteq U$, it follows from the definition of θ_{k_1,k_2} that $\theta_{k_1,k_2}(U) \subseteq \theta_{h_1,h_2}(U)$. The lemma follows by symmetry. □

Definition 2.6. *Assume that R is a (h_1, h_2)-weakly regular $\perp_{h_1,h_2}$-isomorphism. Let us say that $(k_1, k_2) \preceq (h_1, h_2)$ if*

1. *R is a (k_1, k_2)-weakly regular $\perp_{k_1,k_2}$-isomorphism.*
2. *For any $U \in \mathrm{RO}(X_1)$, $x \notin \overline{U}$, there exist $(f_1, f_2) \in R$ and $U' \in \mathrm{RO}(X_1)$ so that $x \in U'$ and that $f_1 = h_1$ on U and $f_1 = k_1$ on U'.*
3. *For any $V \in \mathrm{RO}(X_2)$, $y \notin \overline{V}$, there exist $(g_1, g_2) \in R$ and $V' \in \mathrm{RO}(X_2)$ so that $y \in V'$ and that $g_2 = h_2$ on V and $g_2 = k_2$ on V'.*

Proposition 2.7. *Suppose that R is a (h_1, h_2)-weakly regular $\perp_{h_1,h_2}$-isomorphism. If $(k_1, k_2) \preceq (h_1, h_2)$, then $\theta_{k_1,k_2}(U) = \theta_{h_1,h_2}(U)$ for any $U \in \mathrm{RO}(X_1)$.*

Proof. First we show that $\theta_{k_1,k_2}(U) \subseteq \theta_{h_1,h_2}(U)$ for any $U \in \mathrm{RO}(X_1)$. Set $W := \theta_{k_1,k_2}(U)\backslash\overline{\theta_{h_1,h_2}(U)}$ and let $y \in W$. By assumption, there exist $(g_1, g_2) \in R$ and $V' \in \mathrm{RO}(X_2)$ so that $y \in V'$, $g_2 = h_2$ on $\theta_{h_1,h_2}(U)$ and $g_2 = k_2$ on V'. We may assume that $V' \subseteq \theta_{k_1,k_2}(U)$. Now $g_1 = h_1$ on U and $g_1 = k_1$ on $\theta^{-1}_{k_1,k_2}(V') \subseteq U$. Thus $h_1 = g_1 = k_1$ on $\theta^{-1}_{k_1,k_2}(V')$ and hence $h_2 = k_2$ on V'. In particular, $h_2(y) = k_2(y)$. This proves that $h_2 = k_2$ on W. By Lemma 2.5, $\theta^{-1}_{h_1,h_2}(W) = \theta^{-1}_{k_1,k_2}(W)$. For any $(l_1, l_2) \in R$ such that $\sigma_{h_1}(l_1) \subseteq \theta^{-1}_{k_1,k_2}(W) = \theta^{-1}_{h_1,h_2}(W)$, it follows from Lemma 2.3 that

$$\sigma_{h_2}(l_2) = \theta_{h_1,h_2}(\sigma_{h_1}(l_1)) \subseteq W.$$

By definition of θ_{h_1,h_2}, we can conclude that $\theta_{h_1,h_2}(\theta^{-1}_{k_1,k_2}(W)) \subseteq W$. As a result, $\theta^{-1}_{k_1,k_2}(W) \subseteq \theta^{-1}_{h_1,h_2}(W)$. By choice of W, we also have $\theta^{-1}_{k_1,k_2}(W) \subseteq U$. So $\theta^{-1}_{k_1,k_2}(W) \subseteq \theta^{-1}_{h_1,h_2}(W) \cap U$. But $\theta_{h_1,h_2}(\theta^{-1}_{h_1,h_2}(W) \cap U) = W \cap \theta_{h_1,h_2}(U) = \emptyset$.

Thus $\emptyset = \theta^{-1}_{h_1,h_2}(W) \cap U \supseteq \theta^{-1}_{k_1,k_2}(W)$. Since $\theta^{-1}_{k_1,k_2}$ is a Boolean isomorphism, this means that $W = \emptyset$. As $\theta_{k_1,k_2}(U)$ is a regular open set, this shows that $\theta_{k_1,k_2}(U) \subseteq \theta_{h_1,h_2}(U)$ for any $U \in \mathrm{RO}(X_1)$. Similarly, $\theta^{-1}_{k_1,k_2}(V) \subseteq \theta^{-1}_{h_1,h_2}(V)$ for any $V \in \mathrm{RO}(X_2)$. Given $U \in \mathrm{RO}(X_1)$, set $V = \theta_{h_1,h_2}(U)$. Then $\theta^{-1}_{k_1,k_2}(\theta_{h_1,h_2}(U)) \subseteq U$, i.e., $\theta_{h_1,h_2}(U) \subseteq \theta_{k_1,k_2}(U)$. ☐

3. Homeomorphism induced by a Boolean isomorphism

In the previous section, we have seen that, under some general assumptions, a $\perp_{h_1,h_2}$-isomorphism induces a Boolean isomorphism between $\mathrm{RO}(X_1)$ and $\mathrm{RO}(X_2)$. A question that has been much investigated under the broad heading of "Banach-Stone Theorem" is when does a $\perp_{h_1,h_2}$-isomorphism induce a homeomorphism between X_1 and X_2. In view of §2, this leads to the question of when a Boolean isomorphism between $\mathrm{RO}(X_1)$ and $\mathrm{RO}(X_2)$ induces a homeomorphism between X_1 and X_2. The next result gives an answer to this question. For terminology and facts concerning filterbases, refer to [12]. A Boolean isomorphism $\theta : \mathrm{RO}(X_1) \to \mathrm{RO}(X_2)$ is called a **strong Boolean isomorphism** if for any $U_1, U_2 \in \mathrm{RO}(X_1)$, $\overline{U_1} \cap \overline{U_2} \neq \emptyset$ if and only if $\overline{\theta(U_1)} \cap \overline{\theta(U_2)} \neq \emptyset$. Denote the cardinality of a set A by $|A|$.

Theorem 3.1. *Let X_1, X_2 be Tychonoff spaces and let $\theta : \mathrm{RO}(X_1) \to \mathrm{RO}(X_2)$ be a Boolean isomorphism. Assume that*

(1) *θ is a strong Boolean isomorphism.*
(2) *If $\mathcal{F}$ is a convergent filterbase in $\mathrm{RO}(X_1)$, then $\bigcap_{U\in\mathcal{F}} \overline{\theta(U)} \neq \emptyset$.*
(3) *If $\mathcal{F}$ is a convergent filterbase in $\mathrm{RO}(X_2)$, then $\bigcap_{V\in\mathcal{F}} \overline{\theta^{-1}(V)} \neq \emptyset$.*

Then there is a homeomorphism $\varphi : X_1 \to X_2$ so that $\varphi(U) = \theta(U)$ for all $U \in \mathrm{RO}(X_1)$. Conversely, if such a homeomorphism exists, then conditions (1), (2) and (3) hold.

Proof. It is clear that if $\varphi : X_1 \to X_2$ is a homeomorphism so that $\varphi(U) = \theta(U)$ for all $U \in \mathrm{RO}(X_1)$, then (1), (2) and (3) hold. Conversely, assume that (1), (2) and (3) hold. Denote by $\mathcal{N}_x$ the family of all $U \in \mathrm{RO}(X_1)$ such that $x \in U$. Similarly for $\mathcal{N}_y$, $y \in X_2$. Claim. $|\bigcap_{U\in\mathcal{N}_x} \overline{\theta(U)}| = 1 = |\bigcap_{V\in\mathcal{N}_y} \overline{\theta^{-1}(V)}|$. We only show the second part of the equality. By (3), $\bigcap_{V\in\mathcal{N}_y} \overline{\theta^{-1}(V)} \neq \emptyset$. Suppose that there are distinct $x_1, x_2 \in \bigcap_{V\in\mathcal{N}_y} \overline{\theta^{-1}(V)}$. Then there are $W_1, W_2 \in \mathrm{RO}(X_1)$ such that $x_i \in W_i$, $i = 1, 2$, and $\overline{W_1} \cap \overline{W_2} = \emptyset$. Let $i \in \{1, 2\}$. For all $V \in \mathcal{N}_y$, $x_i \in \overline{\theta^{-1}(V)}$. Thus $W_i \cap \theta^{-1}(V) \neq \emptyset$. Hence, $V \cap \theta(W_i) \neq \emptyset$ for all $V \in \mathcal{N}_y$. It follows that $y \in \overline{\theta(W_1)} \cap \overline{\theta(W_2)}$. This contradicts (1). Hence $|\bigcap_{V\in\mathcal{N}_y} \overline{\theta^{-1}(V)}| = 1$. This completes the proof of the claim. According to the claim, there are functions $\varphi : X_1 \to X_2$ and $\psi : X_2 \to X_1$ so that $\{\varphi(x)\} = \bigcap_{U\in\mathcal{N}_x} \overline{\theta(U)}$ and that $\{\psi(y)\} = \bigcap_{V\in\mathcal{N}_y} \overline{\theta^{-1}(V)}$. Observe that $\varphi(U) \subseteq \overline{\theta(U)}$ for any $U \in \mathrm{RO}(X_1)$. If φ is not continuous at $x_0 \in X_1$, then there exists $V_0 \in \mathrm{RO}(X_2)$, $y_0 := \varphi(x_0) \in V_0$, and $\varphi(U) \not\subseteq \overline{V_0}$ for all $U \in \mathcal{N}_{x_0}$.

In particular, $\theta(U) \not\subseteq V_0$. Thus $\theta(U) \wedge (\neg V_0) \neq \emptyset$, whence $U \wedge (\neg\theta^{-1}(V_0)) \neq \emptyset$ for all $U \in \mathcal{N}_{x_0}$. Therefore $\mathcal{F} := \{U \wedge (\neg\theta^{-1}(V_0)) : U \in \mathcal{N}_{x_0}\}$ is a filterbase in $\mathrm{RO}(X_1)$, which clearly converges to x_0. By (3), $\bigcap_{U\in\mathcal{N}_{x_0}} \overline{\theta(U) \wedge (\neg V_0)} \neq \emptyset$. Let y_1 be a point in the intersection. Then $y_1 \in \bigcap_{U\in\mathcal{N}_{x_0}} \overline{\theta(U)}$ and $y_1 \notin V_0$; in particular, $y_1 \neq y_0$. Therefore, $\bigcap_{U\in\mathcal{N}_{x_0}} \overline{\theta(U)}$ contains at least two points, contrary to the claim. This completes the proof of continuity for φ. By symmetry, ψ is also continuous. Let $x \in X_1$. If $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_{\varphi(x)}$, then $\varphi(x) \in \overline{\theta(U)}$ and hence $\theta(U) \cap V \neq \emptyset$. Thus, $U \cap \theta^{-1}(V) \neq \emptyset$. Hence $x \in \bigcap_{V\in\mathcal{N}_{\varphi(x)}} \overline{\theta^{-1}(V)}$. On the other hand, by definition of ψ, $\psi(\varphi(x)) \in \bigcap_{V\in\mathcal{N}_{\varphi(x)}} \overline{\theta^{-1}(V)}$. It follows by the claim that $\psi(\varphi(x)) = x$. By symmetry, φ and ψ are mutual inverses, Hence φ is a homeomorphism from X_1 onto X_2. Finally, let $U \in \mathrm{RO}(X_1)$. Since $\varphi(U) \subseteq \overline{\theta(U)}$ and $\theta(U) \in \mathrm{RO}(X_2)$, $\varphi(U) \subseteq \theta(U)$. Suppose that $y \in \theta(U)\backslash\overline{\varphi(U)}$. For any $V \in \mathcal{N}_y$, $V \cap \theta(U) \neq \emptyset$ and hence $\theta^{-1}(V) \cap U \neq \emptyset$. Choose $x \in \theta^{-1}(V) \cap U$. Then $\varphi(x) \in \overline{\theta(\theta^{-1}(V) \cap U)} \subseteq \overline{V}$. Thus $\varphi(x) \in \varphi(U) \cap \overline{V}$. This proves that $\varphi(U) \cap \overline{V} \neq \emptyset$ for all $V \in \mathcal{N}_y$. Hence $y \in \overline{\varphi(U)}$, contrary to its choice. Therefore, $\theta(U) \subseteq \overline{\varphi(U)}$ and hence $\theta(U) \subseteq \varphi(U)$ on account of the fact that $\varphi(U) \in \mathrm{RO}(X_2)$. This shows that $\varphi(U) = \theta(U)$ for all $U \in \mathrm{RO}(X_1)$, as required. □

A homeomorphism $\varphi : X_1 \to X_2$ satisfying the conclusion of Theorem 3.1 is said to be **associated with** θ.

Corollary 3.2. *Let X_1, X_2 be Tychonoff spaces and let $\theta : \mathrm{RO}(X_1) \to \mathrm{RO}(X_2)$ be a Boolean isomorphism. Assume that for any $x \in X_1, y \in X_2$, there are $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$ so that $\overline{\theta(U)}$ and $\overline{\theta^{-1}(V)}$ are compact in X_1 and X_2 respectively. Then there is a homeomorphism $\varphi : X_1 \to X_2$ associated with θ if and only if θ is a strong Boolean isomorphism.*

Proof. It suffices to show the "if" statement. By Theorem 3.1 and symmetry, it is enough to show that $\bigcap_{U\in\mathcal{F}} \overline{\theta(U)} \neq \emptyset$ for any filterbase $\mathcal{F} \subseteq \mathrm{RO}(X_1)$ that converges to some $x \in X_1$. Let such a $\mathcal{F}$ be given. By assumption, there exists $U_0 \in \mathcal{N}_x$ so that $\overline{\theta(U_0)}$ is compact. Then $\{\overline{\theta(U \cap U_0)} : U \in \mathcal{F}\}$ is a family of closed subsets of $\overline{\theta(U_0)}$ that has the finite intersection property. Thus $\bigcap_{U\in\mathcal{F}} \overline{\theta(U)} \supseteq \bigcap_{U\in\mathcal{F}} \overline{\theta(U \cap U_0)} \neq \emptyset$, as required. □

The next corollary is immediate.

Corollary 3.3. *Let X_1, X_2 be compact Hausdorff spaces and let $\theta : \mathrm{RO}(X_1) \to \mathrm{RO}(X_2)$ be a Boolean isomorphism. Then there is a homeomorphism $\varphi : X_1 \to X_2$ associated with θ if and only if θ is a strong Boolean isomorphism.*

4. Homeomorphism induced by a $\perp$-isomorphism

We are now ready to use the results from the previous sections to obtain Banach-Stone type theorems for $\perp$-isomorphisms. As before, let X_1, X_2 be Tychonoff spaces and E_1, E_2 be Hausdorff topological spaces. Assume that

$R \subseteq C(X_1,E_1)\times C(X_2,E_2)$ and $(h_1,h_2) \in R$. We are interested in obtaining a homeomorphism $\varphi : X_1 \to X_2$ so that for any $(f_1,f_2) \in R$ and $U \in \mathrm{RO}(X_1)$, $f_1 = h_1$ on U if and only if $f_2 = h_2$ on $\varphi(U)$. Such a homeomorphism φ is said to be **associated with** (R,h_1,h_2). The first application is a known result from [10, Theorem 1.17].

Lemma 4.1. *Suppose that R is (h_1,h_2)-weakly regular and that for any (f_1,f_2), $(g_1,g_2) \in R$, $\overline{[f_1 \neq h_1]} \cap \overline{[g_1 \neq h_1]} = \emptyset$ if and only if $\overline{[f_2 \neq h_2]} \cap \overline{[g_2 \neq h_2]} = \emptyset$. Then R is a $\perp_{h_1,h_2}$-isomorphism.*

Proof. By Proposition 2.1, it suffices to show that R is a $\subseteq_{h_1,h_2}$-isomorphism. Suppose that (f_1,f_2), $(g_1,g_2) \in R$ with $\sigma_{h_1}(f_1) \subseteq \sigma_{h_1}(g_1)$. If $y \notin \overline{[g_2 \neq h_2]}$, then there exists $(k_1,k_2) \in R$ so that $y \in \sigma_{h_2}(k_2)$ and that $\overline{\sigma_{h_2}(k_2)} \cap \overline{[g_2 \neq h_2]} = \emptyset$. Hence we have

$$\begin{aligned} & \overline{[k_2 \neq h_2]} \cap \overline{[g_2 \neq h_2]} = \emptyset \implies \overline{[k_1 \neq h_1]} \cap \overline{[g_1 \neq h_1]} = \emptyset, \\ \implies \ & \overline{[k_1 \neq h_1]} \cap \overline{[f_1 \neq h_1]} = \emptyset \implies \overline{[k_2 \neq h_2]} \cap \overline{[f_2 \neq h_2]} = \emptyset. \end{aligned}$$

In particular, $y \notin \overline{[f_2 \neq h_2]}$. This shows that $\overline{[f_2 \neq h_2]} \subseteq \overline{[g_2 \neq h_2]}$ and hence $\sigma_{h_2}(f_2) \subseteq \sigma_{h_2}(g_2)$. By symmetry, we also have the reverse implication $\sigma_{h_2}(f_2) \subseteq \sigma_{h_2}(g_2) \implies \sigma_{h_1}(f_1) \subseteq \sigma_{h_1}(g_1)$. □

Theorem 4.2. *(Cordeiro) Let X_1, X_2 be compact Hausdorff spaces. Suppose that R is (h_1,h_2)-weakly regular and that for any $(f_1,f_2),(g_1,g_2) \in R$, $\overline{[f_1 \neq h_1]} \cap \overline{[g_1 \neq h_1]} = \emptyset$ if and only if $\overline{[f_2 \neq h_2]} \cap \overline{[g_2 \neq h_2]} = \emptyset$. Then there is a homeomorphism φ associated with (R,h_1,h_2).*

Proof. By Lemma 4.1 and Theorem 2.4, there is a Boolean isomorphism $\theta : \mathrm{RO}(X_1) \to \mathrm{RO}(X_2)$ associated with (R,h_1,h_2). We claim that θ is a strong Boolean isomorphism. Indeed, suppose that $U_1, U_2 \in \mathrm{RO}(X_1)$ and that $\overline{U_1} \cap \overline{U_2} = \emptyset$. There are (f_1^k, f_2^k), (g_1^k, g_2^k) in R, $1 \leq k \leq n$, so that $U_1 \subseteq \bigcup_{k=1}^n \sigma_{h_1}(f_1^k)$, $U_2 \subseteq \bigcup_{k=1}^n \sigma_{h_1}(g_1^k)$ and that $\overline{[f_1^j \neq h_1]} \cap \overline{[g_1^k \neq h_1]} = \emptyset$ for any j,k. By assumption $\overline{[f_2^j \neq h_2]} \cap \overline{[g_2^k \neq h_2]} = \emptyset$. Now $\bigvee_{k=1}^n \sigma_{h_1}(f_1^k) = \operatorname{int} \overline{\bigcup_{k=1}^n \sigma_{h_1}(f_1^k)} \supseteq U_1$. Hence

$$\theta(U_1) \subseteq \theta(\bigvee_{k=1}^n \sigma_{h_1}(f_1^k)) = \bigvee_{k=1}^n \sigma_{h_2}(f_2^k) \subseteq \bigcup_{i=1}^n \overline{[f_2 \neq h_2]}.$$

Hence $\overline{\theta(U_1)} \subseteq \bigcup_{k=1}^n \overline{[f_2^k \neq h_2]}$. Similarly, $\overline{\theta(U_2)} \subseteq \bigcup_{k=1}^n \overline{[g_2^k \neq h_2]}$. Thus $\overline{\theta(U_1)} \cap \overline{\theta(U_2)} = \emptyset$. Allowing for symmetry, this completes the proof of the claim that θ is a strong Boolean isomorphism. By Corollary 3.3, there is a homeomorphism $\varphi : X_1 \to X_2$ associated with θ. Hence φ is associated with (R,h_1,h_2), as required. □

Recall that for any $y \in X_2$, $\mathcal{N}_y$ consists of all $V \in \mathrm{RO}(X_2)$ such that $y \in V$. Similarly for $\mathcal{N}_x$ if $x \in X_1$.

Theorem 4.3. *Let X_1, X_2 be compact Hausdorff spaces and let E_1, E_2 be Hausdorff spaces. Suppose that R is a (h_1, h_2)-weakly regular $\perp_{h_1,h_2}$-isomorphism. Assume that for any $x \in X_1$, $y \in X_2$, there are $(k_1, k_2), (l_1, l_2) \in R$ so that $(k_1, k_2), (l_1, l_2) \preceq (h_1, h_2)$ and that $k_1(x) \neq h_1(x)$, $l_2(y) \neq h_2(y)$. Then there exists a homeomorphism $\varphi : X_1 \to X_2$ associated with (R, h_1, h_2).*

Proof. We aim to apply Theorem 4.2. Let $(f_1, f_2), (g_1, g_2) \in R$ be such that $\overline{[f_1 \neq h_1]} \cap \overline{[g_1 \neq h_1]} = \emptyset$. Assume, if possible, that there exists $y \in \overline{[f_2 \neq h_2]} \cap \overline{[g_2 \neq h_2]}$. Use Proposition 2.2 to find a Boolean isomorphism $\theta = \theta_{h_1,h_2} : \mathrm{RO}(X_1) \to \mathrm{RO}(X_2)$ associated with (R, h_1, h_2). For any $V \in \mathcal{N}_y$, $V \cap \sigma_{h_2}(f_2) \neq \emptyset$ and hence $\theta^{-1}(V) \cap \sigma_{h_1}(f_1) \neq \emptyset$. By compactness of X_1, there exists $x \in \overline{[f_1 \neq h_1]} \cap \bigcap\{\overline{\theta^{-1}(V)} : V \in \mathcal{N}_y\}$. By assumption, there exists $(l_1, l_2) \in R$, $(l_1, l_2) \preceq (h_1, h_2)$ so that $l_2(y) \neq h_2(y)$. Since $x \notin \overline{\sigma_{h_1}(g_1)}$, there exist $(p_1, p_2) \in R$ and $U' \in \mathcal{N}_x$ so that $p_1 = h_1$ on $\sigma_{h_1}(g_1)$ and $p_1 = l_1$ on U'. Since $\theta_{h_1,h_2} = \theta_{l_1,l_2}$ by Proposition 2.7, $p_2 = h_2$ on $\sigma_{h_2}(g_2)$ and $p_2 = l_2$ on $\theta(U')$. By continuity of p_2, $p_2 = h_2$ on $\overline{\sigma_{h_2}(g_2)} = \overline{[g_2 \neq h_2]} \ni y$. On the other hand, if $V \in \mathcal{N}_y$, then $x \in \overline{\theta^{-1}(V)}$ and hence $U' \cap \theta^{-1}(V) \neq \emptyset$; so $\theta(U') \cap V \neq \emptyset$. Hence $y \in \overline{\theta(U')}$. Thus we also have $p_2(y) = l_2(y)$. This is impossible since $l_2(y) \neq h_2(y)$. We have shown that $\overline{[f_1 \neq h_1]} \cap \overline{[g_1 \neq h_1]} = \emptyset$ implies $\overline{[f_2 \neq h_2]} \cap \overline{[g_2 \neq h_2]} = \emptyset$. The reverse implication follows by symmetry. The theorem now follows from Theorem 4.2. □

Given a Tychonoff space X, denote its Stone-Čech compactification by βX. If X, E are Tychonoff spaces, then every $f \in C(X, E)$ extends uniquely to a $f^\beta \in C(\beta X, \beta E)$.

Definition 4.4. *Assume that R is a (h_1, h_2)-weakly regular $\perp_{h_1,h_2}$-isomorphism. Let us say that $(k_1, k_2) \ll (h_1, h_2)$ if*

1. *R is a (k_1, k_2)-weakly regular $\perp_{k_1,k_2}$-isomorphism.*
2. *for any U_1, U_2 in $\mathrm{RO}(X_1)$ so that $\overline{U_1}^{\beta X_1} \cap \overline{U_2}^{\beta X_1} = \emptyset$, there exists $(f_1, f_2) \in R$ so that $f_1 = h_1$ on U_1 and $f_1 = k_1$ on U_2.*
3. *for any V_1, V_2 in $\mathrm{RO}(X_2)$ so that $\overline{V_1}^{\beta X_2} \cap \overline{V_2}^{\beta X_2} = \emptyset$, there exists $(g_1, g_2) \in R$ so that $g_2 = h_2$ on V_1 and $g_2 = k_2$ on V_2.*

Theorem 4.5. *Let X_1, X_2, E_1, E_2 be Tychonoff spaces and let R be a (h_1, h_2)-weakly regular $\perp_{h_1,h_2}$-isomorphism. Assume that for any $x \in \beta X_1$, $y \in \beta X_2$, there are (k_1, k_2), $(l_1, l_2) \in R$ so that $(k_1, k_2), (l_1, l_2) \ll (h_1, h_2)$ and that $k_1^\beta(x) \neq h_1^\beta(x)$, $l_2^\beta(y) \neq h_2^\beta(y)$. Then there exists a homeomorphism $\varphi^\beta : \beta X_1 \to \beta X_2$ so that for any $(f_1, f_2) \in R$ and any $U \in \mathrm{RO}(\beta X_1)$, $f_1 = h_1$ on $U \cap X_1$ if and only if $f_2 = h_2$ on $\varphi^\beta(U) \cap X_2$.*

Proof. Every $f_i \in C(X_i, E_i)$ extends uniquely to a $f_i^\beta \in C(\beta X_i, \beta E_i)$. Define a relation $R^\beta \subseteq C(\beta X_1, \beta E_1) \times C(\beta X_2, \beta E_2)$ by

$$R^\beta = \{(f_1^\beta, f_2^\beta) : (f_1, f_2) \in R\}.$$

We claim that R^β is a (h_1^β, h_2^β)-weakly regular $\perp_{h_1^\beta, h_2^\beta}$-isomorphism. Let (f_1^β, f_2^β), $(g_1^\beta, g_2^\beta) \in R^\beta$ be such that $f_1^\beta \perp_{h_1^\beta} g_1^\beta$. Then $f_1 \perp_{h_1} g_1$ and thus $f_2 \perp_{h_2} g_2$.

If $O = [f_2^\beta \neq h_2^\beta] \cap [g_2^\beta \neq h_2^\beta] \neq \emptyset$, then $[f_2 \neq h_2] \cap [g_2 \neq h_2] = O \cap X_2 \neq \emptyset$, a contradiction. Hence $f_2^\beta \perp_{h_2^\beta} g_2^\beta$. Similarly, $f_1^\beta \perp_{h_1^\beta} g_1^\beta$ if $f_2^\beta \perp_{h_2^\beta} g_2^\beta$. This shows that R^β is a $\perp_{h_1^\beta, h_2^\beta}$-isomorphism. To see that R^β is (h_1^β, h_2^β)-weakly regular, let $U \in \mathrm{RO}(\beta X_i)$. Then $U \cap X_i$ is nonempty and open in X_i. There exists $(f_1, f_2) \in R$ so that $\emptyset \neq \sigma_{h_i}(f_i) \subseteq U \cap X_i$. Therefore, $U^c \cap X_i \subseteq [f_i = h_i] \subseteq [f_i^\beta = h_i^\beta]$. Since U is regular open,

$$U^c = \overline{\mathrm{int}_{\beta X_i}(U^c)}^{\beta X_i} = \overline{(\mathrm{int}_{\beta X_i}(U^c)) \cap X_i}^{\beta X_i} \subseteq \overline{U^c \cap X_i}^{\beta X_i} \subseteq [f_i^\beta = h_i^\beta].$$

Therefore,

$$\emptyset \neq \sigma_{h_i}(f_i) \subseteq \sigma_{h_i^\beta}(f_i^\beta) = \mathrm{int}_{\beta X_i} \overline{[f_i^\beta \neq h_i^\beta]}^{\beta X_i} \subseteq \mathrm{int}_{\beta X_i} \overline{U}^{\beta X_i} = U.$$

So R^β is (h_1^β, h_2^β)-weakly regular. Let (k_1, k_2) be given by the assumption. We claim that $(k_1^\beta, k_2^\beta) \preceq (h_1^\beta, h_2^\beta)$. It suffices to check conditions (2) and (3) in the definition of "$\preceq$". Let $U_1 \in \mathrm{RO}(\beta X_1)$, $x \notin \overline{U_1}^{\beta X_1}$. There exists $U_2 \in \mathrm{RO}(\beta X_1)$ such that $x \in U_2$, $\overline{U_1}^{\beta X_1} \cap \overline{U_2}^{\beta X_1} = \emptyset$. Set $V_i = \mathrm{int}\, \overline{U_i \cap X_1} \in \mathrm{RO}(X_1)$, $i = 1, 2$. Then $\overline{V_1}^{\beta X_1} \cap \overline{V_2}^{\beta X_1} = \emptyset$. By assumption, $(k_1, k_2) \ll (h_1, h_2)$. Thus there exists $(f_1, f_2) \in R$ so that $f_1 = h_1$ on V_1 and $f_1 = k_1$ on V_2. Hence $f_1^\beta = h_1^\beta$ on $\overline{V_1}^{\beta X_1} \supseteq U_1$. Similarly, $f_1^\beta = k_1^\beta$ on $U_2 \ni x$. This verifies condition (2) in the definition of $(k_1^\beta, k_2^\beta) \preceq (h_1^\beta, h_2^\beta)$. Condition (3) can be shown in the same way. This shows that $(k_1^\beta, k_2^\beta) \preceq (h_1^\beta, h_2^\beta)$. Similarly, $(l_1^\beta, l_2^\beta) \preceq (h_1^\beta, h_2^\beta)$. Apply Theorem 4.3 to R^β to obtain a homeomorphism $\varphi^\beta : \beta X_1 \to \beta X_2$ associated with $(R^\beta, h_1^\beta, h_2^\beta)$. The conclusion of the theorem follows easily. $\square$

Remark. In the notation of Theorem 4.5, suppose that V is an open set in βX_1. Let $U = \mathrm{int}_{\beta X_1} \overline{V}^{\beta X_1}$. Then $U \in \mathrm{RO}(\beta X_1)$ and $V \subseteq U \subseteq \overline{V}^{\beta X_1}$. If $(f_1, f_2) \in R$, then

$$\begin{aligned} & f_1 = h_1 \text{ on } V \cap X_1 \iff f_1 = h_1 \text{ on } U \cap X_1 \\ \iff \quad & f_2 = h_2 \text{ on } \varphi^\beta(U) \cap X_2 \iff f_2 = h_2 \text{ on } \varphi^\beta(V) \cap X_2. \end{aligned}$$

Corollary 4.6. *Let X_1, X_2, E_1, E_2 be Tychonoff spaces so that X_1, X_2 are first countable. Suppose that R is a (h_1, h_2)-weakly regular $\perp_{h_1,h_2}$-isomorphism. Assume that for any $x \in \beta X_1$, $y \in \beta X_2$, there are $(k_1, k_2), (l_1, l_2) \in R$ so that $(k_1, k_2), (l_1, l_2) \ll (h_1, h_2)$ and that $k_1^\beta(x) \neq h_1^\beta(x)$, $l_2^\beta(y) \neq h_2^\beta(y)$. Then there exist a homeomorphism $\varphi : X_1 \to X_2$ associated with (R, h_1, h_2).*

Proof. Obtain a homeomorphism $\varphi^\beta : \beta X_1 \to \beta X_2$ by Theorem 4.5. We claim that if $x \in X_1$, then βX_1 has a countable basis at x. Indeed, let (U_n) be a countable basis at x in X_1. For each n, choose an open set V_n in βX_1 so that $U_n = V_n \cap X_1$. Suppose that W is an open neighborhood of x in βX_1. There exists n such that $U_n \subseteq W \cap X_1$. If $V_n \backslash \overline{W}^{\beta X_1} \neq \emptyset$, then $(V_n \backslash \overline{W}^{\beta X_1}) \cap X_1 \neq \emptyset$. Hence $U_n \backslash \overline{W}^{\beta X_1} \neq \emptyset$, contradicting the choice of n. This shows that for any open neighborhood W of x in βX_1, there exists n so that $x \in V_n \subseteq \overline{W}^{\beta X_1}$.

Since βX_1 is a regular topological space, we conclude that (V_n) is a basis at x in βX_1. As φ^β is a homeomorphism, βX_2 has a countable basis at $\varphi^\beta(x)$. However, by [16, Corollary 9.6], no point in $\beta X_2 \backslash X_2$ can be a G_δ-point. Thus $\varphi^\beta(x) \in X_2$. Hence $\varphi^\beta(X_1) \subseteq X_2$. By symmetry, φ^β maps X_1 onto X_2. Let $\varphi = \varphi^\beta|_{X_1}$. Then φ is a homeomorphism associated with (R, h_1, h_2). (See the Remark before the corollary.) □

Recall that π_i is the projection from $C(X_1, E_1) \times C(X_2, E_2)$ onto the ith factor, $i = 1, 2$.

Corollary 4.7. *Let X_1, X_2 be Tychonoff spaces and let E_1, E_2 be infinite convex sets in Hausdorff topological vector spaces. Suppose that $R \subseteq C(X_1, E_1) \times C(X_2, E_2)$ so that $\pi_i(R) = C(X_i, E_i)$. Assume that there are (k_1, k_2), (k_1', k_2'), (l_1, l_2), $(l_1', l_2') \in R$ so that R is a $\perp_{k_1,k_2}$-, $\perp_{k_1',k_2'}$-, $\perp_{l_1,l_2}$- and $\perp_{l_1',l_2'}$-isomorphism; k_1, k_1' are distinct constant functions, as are l_2, l_2'. Then βX_1 and βX_2 are homeomorphic. If, in addition, either both X_1, X_2 are compact or both are first countable, then X_1, X_2 are homeomorphic.*

Proof. Every Hausdorff topological vector space is Tychonoff; see, e.g., [27, p16, 1.4]. Thus, E_1, E_2 are Tychonoff spaces. Since $\pi_i(R) = C(X_i, E_i)$, it is clear the R is (f_1, f_2)-weakly regular for any $(f_1, f_2) \in R$. We will attempt to apply Theorem 4.5 with $(h_1, h_2) = (k_1, k_2)$. Let $x_0 \in \beta X_1$. Then $(k_1'^\beta(x_0) \neq k_1^\beta(x_0)$. We claim that $(k_1', k_2') \ll (k_1, k_2)$. By assumption, condition (1) in Definition 4.4 holds for (k_1', k_2'). Assume that $U_1, U_2 \in \mathrm{RO}(X_1)$ with $\overline{U_1}^{\beta X_1} \cap \overline{U_2}^{\beta X_2} = \emptyset$. There exists a continuous function $g : X_1 \to [0, 1]$ so that $g(x) = 0$ on U_1 and $g(x) = 1$ on U_2. Define

$$f_1 : X_1 \to E_1 \text{ by } f_1(x) = (1 - g(x))k_1(x) + g(x)k_1'(x).$$

Then $f_1 \in \pi_1(R)$, $f_1 = k_1$ on U_1 and $f_1 = k_1'$ on U_2. This completes the proof of condition (2) in Definition 4.4. Condition (3) is verified similarly. Analogously, for any $y_0 \in \beta X_2$, either ${}^{\beta}_{1}(y_0) \neq k_2^\beta(y_0)$ or $({}'^{\beta}_{1}(y_0) \neq k_2^\beta(y_0)$. Moreover, both (l_1, l_2) and $(l_1', l_2') \ll (k_1, k_2)$. By Theorem 4.5, βX_1 and βX_2 are homeomorphic. In particular X_1, X_2 are homeomorphic if both are compact. If both X_1, X_2 are first countable, then they are homeomorphic by Corollary 4.6. □

5. Application: Order isomorphisms

In this part, we give applications of the results in the preceding sections to order isomorphisms. Order isomorphisms between spaces of real valued functions have been studied in e.g., [7, 8, 9, 23]. A classical result of Kaplansky [21] dealt with order isomorphisms between spaces of functions taking values in totally ordered spaces with the order topology. We generalize this result to functions taking values in topological lattices. A **lattice** is a partially ordered set E so that the least upper bound $u \vee v$ and the greatest lower bound $u \wedge v$ exist for any $u, v \in E$. A **topological lattice** is a lattice with a topology so that the lattice operations $(u, v) \mapsto u \wedge v, u \vee v$ are continuous on $E \times E$. If X is a Tychonoff space and E is a topological lattice, then $C(X, E)$ is a lattice under the pointwise order. A subset $A(X, E)$ of $C(X, E)$ is said to be **E-normal** if for any disjoint closed subsets K_1, K_2 of X and any $e_1, e_2 \in E$, there exists $f \in A(X, E)$ so that $f(x) = e_i$ if $x \in K_i$. A function $f : X \to E$ is **order bounded** if there are $e_1, e_2 \in E$ so that $e_1 \leq f(x) \leq e_2$ for all $x \in X$.

Lemma 5.1. *Assume that $A(X, E)$ is a E-normal sublattice of $C(X, E)$ that consists of order bounded functions. Let $h, k \in A(X, E)$ and let H, K be disjoint closed sets in X. Then there exists $g \in A(X, E)$ so that $g = h$ on H, $g = k$ on K.*

Proof. Let h, k and H, K be as in the statement of the lemma. There are $e_1, e_2 \in E$ so that $e_1 \leq h, k \leq e_2$ on X. By E-normality, there are $f_1, f_2 \in A(X, E)$ so that

$$f_1 = \begin{cases} e_2 & \text{on } H \\ e_1 & \text{on } K \end{cases} \quad \text{and} \quad f_2 = \begin{cases} e_1 & \text{on } H \\ e_2 & \text{on } K \end{cases}.$$

Set $g = (f_2 \vee h) \wedge (f_1 \vee k)$. Then $g \in A(X, E)$. It is easy to see that $g = h$ on H and $g = k$ on K. □

For the rest of the section, let X_i be a Tychonoff space and E_i be a non-singleton Hausdorff topological lattice, $i = 1, 2$. Assume that $A(X_i, E_i)$ is a E_i-normal sublattice of $C(X_i, E_i)$ consisting of order bounded functions and let $R : A(X_1, E_i) \to A(X_2, E_2)$ be an **order isomorphism**; that is, R is a bijection so that $f_1 \leq g_1 \iff Rf_1 \leq Rg_1$ for any $f_1, g_1 \in A(X_1, E_1)$. In this case, R preserves the lattice operations: if $f_1, g_1 \in A(X_1, E_1)$, then $R(f_1 \wedge g_1) = Rf_1 \wedge Rg_1$. Of course we may view R as the relation

$$\{(f_1, Rf_1) : f_1 \in A(X_1, E_1)\} \subseteq C(X_1, E_1) \times C(X_2, E_2).$$

If π_i is the projection from $C(X_1, E_1) \times C(X_2, E_2)$ onto the i-th component, then $\pi_i(R) = A(X_i, E_i)$.

Proposition 5.2. *R is a $\perp_{h_1, Rh_1}$-isomorphism for any $h_1 \in A(X_1, E_1)$.*

Proof. Let $h_1 \in A(X_1, E_1)$. Consider $h_1, g_1 \in A(X_1, E_1)$ so that $f_1 \perp_{h_1} g_1$. If $f_1, g_1 \geq h_1$, then $f_1 \wedge g_1 = h_1$ and hence $Rf_1 \wedge Rg_1 = Rh_1$; whence $Rf_1 \perp_{Rh_1} Rg_1$. Similarly, $Rf_1 \perp_{Rh_1} Rg_1$ if $f_1 \perp_{h_1} g_1$ and $f_1, g_1 \leq h_1$. Claim.

If $f_1, g_1 \in A(X_1, E_1)$, $f_1 \perp_{h_1} g_1$ and $f_1 \geq h_1 \geq g_1$, then $Rf_1 \perp_{Rh_1} Rg_1$. Otherwise, there exists $y \in X_2$ so that $Rf_1(y) > Rh_1(y) > Rg_1(y)$. (Here $u, v \in E_2$, $u > v$ means $u \geq v$ and $u \neq v$.) Let $U = \sigma_{Rg_1}(Rh_1) \in \mathrm{RO}(X_2)$. By Lemma 5.1, there exists $k_2 \in A(X_2, E_2)$ so that $k_2(y) = Rf_1(y)$ and $k_2 = Rg_1 = Rh_1$ on U^c. Replace k_2 by $(k_2 \vee Rh_1) \wedge Rf_1$ if necessary to assume additionally that $Rh_1 \leq k_2 \leq Rf_1$. Let $k_1 = R^{-1}k_2$. Since $g_1 \leq h_1 \leq k_1$, we have $\sigma_{g_1}(h_1) \subseteq \sigma_{g_1}(k_1)$. We will show that $\sigma_{g_1}(k_1) \subseteq \overline{\sigma_{g_1}(h_1)}$. Suppose that it does not hold. There exists a nonempty open set W contained in $\sigma_{g_1}(k_1)$ so that $\overline{W} \cap \overline{\sigma_{g_1}(h_1)} = \emptyset$. By Lemma 5.1 again, there exists $l_1 \in A(X_1, E_1)$ so that $l_1 = g_1$ on $\overline{\sigma_{g_1}(h_1)}$ and $l_1 = k_1$ on W. Replace l_1 by $l_1 \vee g_1$ if necessary to assume that $l_1 \geq g_1$. (Note that $g_1 \leq h_1 \leq k_1 \leq f_1$.) Then $l_1, h_1 \geq g_1$ and $l_1 \perp_{g_1} h_1$. Hence $Rl_1 \perp_{Rg_1} Rh_1$ by the paragraph before the claim. Thus

$$\sigma_{Rg_1}(Rl_1) \cap \sigma_{Rg_1}(k_2) \subseteq \sigma_{Rg_1}(Rl_1) \cap U = \sigma_{Rg_1}(Rl_1) \cap \sigma_{Rg_1}(Rh_1) = \emptyset.$$

Hence $Rl_1 \perp_{Rg_1} k_2$. Since $l_1, k_1 \geq g_1$ as well, $l_1 \perp_{g_1} k_1$. But $l_1 = k_1$ on W. Hence $k_1 = g_1$ on $W \subseteq \sigma_{g_1}(k_1)$. So we must have $W = \emptyset$, contrary to its choice. Thus $\sigma_{g_1}(k_1) \subseteq \overline{\sigma_{g_1}(h_1)}$ and hence $\sigma_{g_1}(k_1) \subseteq \sigma_{g_1}(h_1)$. By assumption, $\sigma_{h_1}(f_1) \cap \sigma_{g_1}(h_1) = \sigma_{h_1}(f_1) \cap \sigma_{h_1}(g_1) = \emptyset$. Therefore, $\sigma_{g_1}(k_1) \cap \sigma_{h_1}(f_1) = \emptyset$. If $x \in \sigma_{g_1}(k_1)$, then $x \notin \sigma_{h_1}(f_1)$ and hence $f_1(x) = h_1(x)$. So $k_1(x) = h_1(x)$ since $f_1 \geq k_1 \geq h_1$. On the other hand, if $x \notin \sigma_{g_1}(k_1)$, then $g_1(x) = k_1(x)$ and hence $k_1(x) = h_1(x)$ since $k_1 \geq h_1 \geq g_1$. Combining the two cases yields $k_1 = h_1$ and hence $k_2 = Rh_1$, which is impossible since they differ at y. This completes the proof of the claim. Finally, for any $f_1, g_1 \in A(X_1, E_1)$ with $f_1 \perp_{h_1} g_1$, it follows from the first paragraph and the claim that

$$Rf_1 \blacklozenge Rh_1 \perp_{Rh_1} Rg_1 \lozenge Rh_1, \tag{5.1}$$

where each of the symbols $\blacklozenge$ and $\lozenge$ is chosen from the set $\{\vee, \wedge\}$. Observe that

$$[Rf_1 \neq Rh_1] = [(Rf_1 \vee Rh_1) \neq Rh_1] \cup [(Rf_1 \wedge Rh_1) \neq Rh_1], \text{ and}$$
$$[Rg_1 \neq Rh_1] = [(Rg_1 \vee Rh_1) \neq Rh_1] \cup [(Rg_1 \wedge Rh_1) \neq Rh_1].$$

Therefore, it follows from (5.1) that $[Rf_1 \neq Rh_1] \cap [Rg_1 \neq Rh_1] = \emptyset$, i.e., $Rf_1 \perp_{Rh_1} Rg_1$. By symmetry, $Rf_1 \perp_{Rh_1} Rg_1$ implies $f_1 \perp_{h_1} g_1$. This completes the proof of the proposition. □

Proposition 5.3. *R is (h_1, h_2)-weakly regular for any $(h_1, h_2) \in R$.*

Proof. Let $(h_1, h_2) \in R$ and let $x \in X_1$. Since $|E_1| > 1$, there exists $e \in E_1$ so that $e \neq h_1(x)$. As $A(X_1, E_1)$ is E_1-normal, there exists $f_1 \in \pi_1(R)$ so that $f_1(x) = e$. Let U be an open neighborhood of x in X_1. Choose an open neighborhood V of x so that $\overline{V} \subseteq U$. By Lemma 5.1, there exists $g_1 \in \pi_1(E)$ so that $g_1(x) = f_1(x)$ and $g_1 = h_1$ on V^c. Then $x \in \sigma_{h_1}(g_1) \subseteq \overline{V} \subseteq U$. This proves that $A(X_1, E_1)$ is h_1-weakly regular. Similarly $A(X_2, E_2)$ is h_2-weakly regular. □

The following theorem generalizes the classical theorem of Kaplansky [21] and also [10, Theorem 3.9].

Theorem 5.4. *Let X_1, X_2 be Tychonoff spaces and let E_1, E_2 be Tychonoff topological lattices. Suppose that $A(X_i, E_i)$ is a E_i-normal sublattice of $C(X_i, E_i)$ consisting of order bounded functions, $i = 1, 2$, and that $R : A(X_1, E_1) \to A(X_2, E_2)$ is an order isomorphism. Then there is a homeomorphism $\varphi^\beta : \beta X_1 \to \beta X_2$ such that for any $f_1, h_1 \in A(X_1, E_1)$ and U in* RO(βX_1), *$f_1 = h_1$ on $U \cap X_1$ if and only if $Rf_1 = Rh_1$ on $\varphi^\beta(U) \cap X_2$.*

Proof. Let $h_1 \in A(X_1, E_1)$. By Propositions 5.2 and 5.3, R is a (h_1, Rh_1)-weakly regular $\perp_{h_1, Rh_1}$-isomorphism. Let $x \in \beta X_1$. We wish to obtain $k_1 \in A(X_1, E_1)$ so that $(k_1, Rk_1) \ll (h_1, Rh_1)$ and $k_1^\beta(x) \neq h_1^\beta(x)$. Once this is shown, a similar statement for any $y \in \beta X_2$ can be proved in the same way. The theorem then follows from Theorem 4.5. Now we proceed to prove the assertion in the previous paragraph. Choose an open neighborhood U of x in βX_1 and let $e \in E_1 \backslash \{h_1^\beta(x)\}$. Since $A(X_1, E_1)$ is E_1-normal, there exists $k_1 \in A(X_1, E_1)$ so that $k_1 = e$ on $\overline{U \cap X_1}$. (The second set in the definition of E_1-normality can be taken to be $\emptyset$.) In particular, $k_1^\beta(x) = e \neq h_1^\beta(x)$. To complete the proof, it suffices to show that $(k_1, Rk_1) \ll (h_1, Rh_1)$. By Proposition 5.3, R is (k_1, Rk_1)-weakly regular. Let U_1, U_2 in $\mathrm{RO}(X_1)$ be such that $\overline{U_1}^{\beta X_1} \cap \overline{U_2}^{\beta X_1} = \emptyset$. Then $\overline{U_1}$ and $\overline{U_2}$ are disjoint closed sets in X_1. By Lemma 5.1, there exists $g_1 \in A(X_1, E_1)$ so that $g_1 = h_1$ on $\overline{U_1} \supseteq U_1$ and $g_1 = k_1$ on $\overline{U_2} \supseteq U_2$. This completes the verification of condition (2) in Definition 4.4. Condition (3) in the definition can be shown similarly. This completes the proof that $(k_1, Rk_1) \ll (h_1, Rh_1)$, as required. □

In Theorem 5.4, if in addition both X_1, X_2 are compact or both are first countable, then they are homeomorphic. Indeed, the claim is obvious if both X_1, X_2 are compact; if both are first countable, the conclusion follows as in Corollary 4.6.

6. Application: Nonvanishing preservers

For $i = 1, 2$, let X_i be a Tychonoff space and let E_i be a Hausdorff space. As above, let $R \subseteq C(X_1, E_1) \times C(X_2, E_2)$ and $(h_1, h_2) \in R$. Set $A(X_i, E_i) = \pi_i(R)$, where π_i is the projection from $C(X_1, E_1) \times C(X_2, E_2)$ onto the i-th component. Say that R is a $\cap^n_{h_1,h_2}$**-isomorphism** if for any $(f_1^k, f_2^k) \in R$, $k = 1, \ldots, n$,

$$\bigcap_{k=1}^{n} [f_1^k = h_1] \neq \emptyset \iff \bigcap_{k=1}^{n} [f_2^k = h_2] \neq \emptyset.$$

A **non-vanishing preserver** is a relation R that is a $\cap^n_{h_1,h_2}$-isomorphism for all $n \in \mathbb{N}$. Characterizations of non-vanishing preservers in various scenarios have been given in [10, 11, 17, 22, 25, 26]. It is clear that if $n > m$, then a $\cap^n_{h_1,h_2}$-isomorphism is a $\cap^m_{h_1,h_2}$-isomorphism. In this section, we prove several general theorems concerning (nonlinear) $\cap^2_{h_1,h_2}$-isomorphisms, generalizing, in particular, recent results from [25]. Say that $A(X_i, E_i)$ is h_i**-fine** if for any nonempty $U \in \mathrm{RO}(X_i)$, there exists $k \in A(X_i, E_i)$ so that $\emptyset \neq [k = h_i] \subseteq U$.

Proposition 6.1. *Assume that $A(X_i, E_i)$ is h_i-fine, $i = 1, 2$. If R is a $\cap^2_{h_1,h_2}$-isomorphism, then it is a $\perp_{h_1,h_2}$-isomorphism.*

Proof. Suppose that there are $(f_1, f_2), (g_1, g_2) \in R$ so that $f_1 \perp_{h_1} g_1$, but $f_2 \not\perp_{h_2} g_2$. Pick $V \in \mathrm{RO}(X_2)$ so that $\emptyset \neq V \subseteq [f_2 \neq h_2] \cap [g_2 \neq h_2]$. Since $A(X_2, E_2)$ is h_2-fine, there exists $(k_1, k_2) \in R$ so that $\emptyset \neq [k_2 = h_2] \subseteq V$. Now $[f_2 = h_2] \cap [k_2 = h_2] = \emptyset$; hence $[f_1 = h_1] \cap [k_1 = h_1] = \emptyset$. Similarly, $[g_1 = h_1] \cap [k_1 = h_1] = \emptyset$. However, $f_1 \perp_{h_1} g_1$ implies that $[f_1 = h_1] \cup [g_1 = h_1] = X_1$. Thus $[k_1 = h_1] = \emptyset$. This implies that $[k_2 = h_2] = \emptyset$, contrary to choice of k_2. Therefore, $f_2 \perp_{h_2} g_2$. The proposition follows by symmetry. □

Following [25], we say that $A(X_i, E_i)$ is h_i-**strongly regular** if given $\emptyset \neq U \in \mathrm{RO}(X_i)$ and $x \in U$, there are (f_1, f_2), $(g_1, g_2) \in R$ so that $x \in [f_i \neq h_i] \subseteq U$, $g_i = h_i$ on a neighborhood of x and that $[f_i = h_i] \cap [g_i = h_i] = \emptyset$. Note that it follows from the conditions in the definition that $\emptyset \neq [g_i = h_i] \subseteq U$. In particular, h_i-strong regularity implies h_i-weak regularity and h_i-fineness. Therefore, the next result follows immediately from Proposition 6.1 and Theorem 2.4.

Corollary 6.2. *Assume that $A(X_i, E_i)$ is h_i-strongly regular, $i = 1, 2$. If R is a $\cap^2_{h_1,h_2}$-isomorphism, then there is a Boolean isomorphism $\theta : \mathrm{RO}(X_1) \to \mathrm{RO}(X_2)$ associated with (R, h_1, h_2).*

Recall that for any $x \in X_1$, $\mathcal{N}_x = \{U \in \mathrm{RO}(X_1) : x \in U\}$. If $y \in X_2$, $\mathcal{N}_y$ is defined similarly.

Proposition 6.3. *Assume that $A(X_i, E_i)$ is h_i-strongly regular, $i = 1, 2$. Let R be a $\cap^2_{h_1,h_2}$-isomorphism, with associated Boolean isomorphism $\theta : \mathrm{RO}(X_1) \to \mathrm{RO}(X_2)$. Suppose that for any filterbase $\mathcal{F} \subseteq \mathrm{RO}(X_2)$ that converges in X_2, $\bigcap_{V \in \mathcal{F}} \overline{\theta^{-1}(V)} \neq \emptyset$. Then $\overline{\theta(U_1)} \cap \overline{\theta(U_2)} = \emptyset$ for any $U_1, U_2 \in \mathrm{RO}(X_1)$ such that $\overline{U_1} \cap \overline{U_2} = \emptyset$.*

Proof. Let $U_1, U_2 \in \mathrm{RO}(X_1)$ such that $\overline{U_1} \cap \overline{U_2} = \emptyset$. If there exists $y \in \overline{\theta(U_1)} \cap \overline{\theta(U_2)}$, then for all $W \in \mathcal{N}_y$, $W \cap \theta(U_1) \neq \emptyset \neq W \cap \theta(U_2)$. Hence $\theta^{-1}(W) \cap U_1 \neq \emptyset \neq \theta^{-1}(W) \cap U_2$. Let $\mathcal{F} = \{W \cap \theta(U_1) : W \in \mathcal{N}_y\}$. Then $\mathcal{F}$ is a filterbase in $\mathrm{RO}(X_2)$ that is convergent to y. By assumption, $C_1 := \bigcap_{W \in \mathcal{N}_y} \overline{\theta^{-1}(W) \cap U_1} \neq \emptyset$. Similarly, $C_2 := \bigcap_{W \in \mathcal{N}_y} \overline{\theta^{-1}(W) \cap U_2} \neq \emptyset$. Pick points u_1 and u_2 from C_1 and C_2 respectively. Since $u_1 \neq u_2$, there are $O_1, O_2 \in \mathrm{RO}(X_1)$ so that $u_1 \in O_1, u_2 \in O_2$ and $\overline{O_1} \cap \overline{O_2} = \emptyset$. By h_i-strong regularity, there are $(f_1, f_2), (g_1, g_2) \in R$ and an open neighborhood O of u_1 so that

$$u_1 \in [f_1 \neq h_1] \subseteq O_1, g_1 = h_1 \text{ on } O \text{ and } [f_1 = h_1] \cap [g_1 \cap h_1] = \emptyset,$$

where O is some regular neighborhood of u_1. For any $W \in \mathcal{N}_y$, $u_1 \in \overline{\theta^{-1}(W)}$. So $O \cap \theta^{-1}(W) \neq \emptyset$. Hence, $\theta(O) \cap W \neq \emptyset$. Thus $y \in \overline{\theta(O)}$. Similarly, $y \in \overline{\theta(O_2)}$. Since $g_1 = h_1$ on O, $g_2 = h_2$ on $\overline{\theta(O)}$. In particular, $g_2(y) = h_2(y)$. On the other hand, $\sigma_{h_1}(f_1) \subseteq O_1$ and thus $\sigma_{h_1}(f_1) \cap O_2 = \emptyset$; whence $f_1 = h_1$ on O_2. It follows that $f_2 = h_2$ on $\overline{\theta(O_2)}$ and hence $f_2(y) = h_2(y)$. Therefore, $y \in [f_2 = h_2] \cap [g_2 = h_2]$. This contradicts the fact that R is a $\cap^2_{h_1,h_2}$-isomorphism and that $[f_1 = h_1] \cap [g_1 \cap h_1] = \emptyset$. □

Lemma 6.4. *Let R be a $\cap^2_{h_1,h_2}$-isomorphism so that $A(X_i, E_i)$ is h_i-strongly regular, $i = 1, 2$. Assume that R has an associated homeomorphism $\varphi : X_1 \to X_2$. For any $x \in X_1$ and $(f_1, f_2) \in R$, $f_1(x) = h_1(x) \iff f_2(\varphi(x)) = h_2(\varphi(x))$.*

Proof. Suppose that $(f_1, f_2) \in R$, $f_1(x) = h_1(x)$ and that $f_2(\varphi(x)) \neq h_2(\varphi(x))$. Let $U \in \mathrm{RO}(X_2)$ be such that $\varphi(x) \in U \subseteq [f_2 \neq h_2]$. Since $A(X_2, E_2)$ is h_2-strongly regular, there exist $(g_1, g_2) \in R$ and $V \in \mathrm{RO}(X_2)$ so that $\varphi(x) \in V$,

$g_2 = h_2$ on V and that $[g_2 = h_2] \subseteq U$. Then $g_1 = h_1$ on $\varphi^{-1}(V) \ni x$. However, $[f_2 = h_2] \cap [g_2 = h_2] = \emptyset$ and thus $[f_1 = h_1] \cap [g_1 = h_1] = \emptyset$. This is impossible since the latter set contains x. This proves that $f_1(x) = h_1(x)$ implies $f_2(\varphi(x)) \neq h_2(\varphi(x))$. The reverse implication follows by symmetry. □

The next result generalizes [25, Theorem 2.2].

Theorem 6.5. *Assume that for $i = 1, 2$, X_i is compact Hausdorff and $A(X_i, E_i)$ is h_i-strongly regular. Then R is a $\cap^2_{h_1,h_2}$-isomorphism if and only if there exists a homeomorphism $\varphi : X_1 \to X_2$ so that for any $(f_1, f_2) \in R$ and any $x \in X_1$, $f_1(x) = h_1(x) \iff f_2(\varphi(x)) = h_2(\varphi(x))$.*

Proof. Obviously, if such a φ exists, then R is a $\cap^2_{h_1,h_2}$-isomorphism. Conversely, assume that R is a $\cap^2_{h_1,h_2}$-isomorphism. By Corollary 6.2, there is a Boolean isomorphism $\theta : \mathrm{RO}(X_1) \to \mathrm{RO}(X_2)$ associated with (R, h_1, h_2). Let $\mathcal{F} \subseteq \mathrm{RO}(X_2)$ be a convergent filterbase. Then $\mathcal{F}$ has the finite intersection property and hence $\bigcap_{V \in \mathcal{F}} \overline{\theta^{-1}(V)} \neq \emptyset$. By the same argument $\bigcap_{V \in \mathcal{F}} \overline{\theta(V)} \neq \emptyset$ if $\mathcal{F} \subseteq \mathrm{RO}(X_1)$ is a convergent filterbase. By Proposition 6.3, θ is a strong Boolean isomorphism. It now follows from Corollary 3.2 that there is a homeomorphism $\varphi : X_1 \to X_2$ associated with θ and hence with (R, h_1, h_2). The final conclusion of the theorem follows from Lemma 6.4. □

In the absence of compactness of X_i, we make a stronger assumption on $A(X_i, E_i)$. Say that $A(X_i, E_i)$ **identifies points in X_i precisely (with respect to h_i)** if for any $x \in X_i$, there exists $f_i \in A(X_i, E_i)$ so that $[f_i = h_i] = \{x\}$. See [25].

Lemma 6.6. *Suppose that R is a $\cap^2_{h_1,h_2}$-isomorphism, $A(X_1, E_1)$ is h_1-strongly regular and identifies points in X_1 precisely. Then $\bigcap_{U \in \mathcal{N}_x} \overline{\theta(U)} \neq \emptyset$ for any $x \in X_1$.*

Proof. Let $x \in X_1$. There exists $f_1 \in A(X_1, E_1)$ so that $[f_1 = h_1] = \{x\}$. Suppose that $\bigcap_{U \in \mathcal{N}_x} \overline{\theta(U)} = \emptyset$. Let $y \in X_2$. There exists $U \in \mathcal{N}_x$ so that $y \notin \overline{\theta(U)}$. Choose $V \in \mathcal{N}_y$ so that $\overline{V} \cap \overline{\theta(U)} = \emptyset$. As $A(X_1, E_1)$ is h_1-strongly regular, there exists $(g_1, g_2) \in R$ so that $x \in [g_1 \neq h_1] \subseteq U$. Since $g_1 = h_1$ on $\neg U$, $g_2 = h_2$ on $\neg\theta(U)$. Thus $g_2(y) = h_2(y)$. However, $[f_1 = h_1] \cap [g_1 = h_1] = \emptyset$. Hence $[f_2 = h_2] \cap [g_2 = h_2] = \emptyset$. It follows that $f_2(y) \neq h_2(y)$. As $y \in X_2$ is arbitrary, $[f_2 = h_2] = \emptyset$, from which we see that $[f_1 = h_1] = \emptyset$, which is absurd. □

Theorem 6.7. *Assume that for $i = 1, 2$, X_i, E_i are Tychonoff spaces, $A(X_i, E_i)$ is h_i-strongly regular and identifies points in X_i precisely. Then R is a $\cap^2_{h_1,h_2}$-isomorphism if and only if there exists a homeomorphism $\varphi : X_1 \to X_2$ so that for any $(f_1, f_2) \in R$ and any $x \in X_1$, $f_1(x) = h_1(x) \iff f_2(\varphi(x)) = h_2(\varphi(x))$.*

Proof. The "if" part is clear. Conversely, assume that R is a $\cap^2_{h_1,h_2}$-isomorphism. By Corollary 6.2, there is a Boolean isomorphism $\theta : \mathrm{RO}(X_1) \to \mathrm{RO}(X_2)$ associated with (R, h_1, h_2). Let $x \in \mathrm{RO}(X_1)$. By Lemma 6.6, there exists

$y_1 \in \bigcap_{U\in\mathcal{N}_x} \overline{\theta(U)}$. Claim. $\bigcap_{U\in\mathcal{N}_x} \overline{\theta(U)}^{\beta X} \subseteq X_2$. If the claim fails, there is also a $y_2 \in \bigcap_{U\in\mathcal{N}_x} \overline{\theta(U)}^{\beta X_2}\backslash X_2$. Obviously y_1 and y_2 are distinct points in βX_2. Choose open sets V_1, V_2 in βX_2 so that $y_1 \in V_1$, $y_2 \in V_2$ and that $\overline{V_1}^{\beta X_2} \cap \overline{V_2}^{\beta X_2} = \emptyset$. Let $W_1 = \operatorname{int} \overline{V_1 \cap X_2}$, with interior and closure taken in X_2. Then $y_1 \in W_1 \in \mathrm{RO}(X_2)$. Since $A(X_2, E_2)$ is h_2-strongly regular, there are $(f_1, f_2), (g_1, g_2) \in R$ and $W' \in \mathrm{RO}(X_2)$ so that $y_1 \in [f_2 \neq h_2] \subseteq W_1$, $g_2 = h_2$ on W', $y_1 \in W'$ and that $[f_2 = h_2] \cap [g_2 = h_2] = \emptyset$. If $U \in \mathcal{N}_x$, then $y_1 \in \overline{\theta(U)}$. Hence $W' \cap \theta(U) \neq \emptyset$, which implies that $\theta^{-1}(W') \cap U \neq \emptyset$. This shows that $x \in \overline{\theta^{-1}(W')}$. Moreover, $g_2 = h_2$ on W' implies that $g_1 = h_1$ on $\theta^{-1}(W')$. Thus $g_1(x) = h_1(x)$. On the other hand, set $W_2 = \operatorname{int} \overline{V_2 \cap X_2}$. Then $f_2 = h_2$ on $W_2 \in \mathrm{RO}(X_2)$ and hence $f_1 = h_1$ on $\theta^{-1}(W_2)$. For any $U \in \mathcal{N}_x$, $y_2 \in \overline{\theta(U)}^{\beta X_2}$ and thus $V_2 \cap \theta(U) \neq \emptyset$, which implies that $W_2 \cap \theta(U) \neq \emptyset$. Hence $\theta^{-1}(W_2) \cap U \neq \emptyset$. This proves that $x \in \overline{\theta^{-1}(W_2)}$. Since $f_2 = h_2$ on W_2, $f_1 = h_1$ on $\theta^{-1}(W_2)$. In particular, $f_1(x) = h_1(x)$. It follows from the preceding that $x \in [f_1 = h_1] \cap [g_1 = h_1]$, which implies that $[f_2 = h_2] \cap [g_2 = h_2] \neq \emptyset$, contradicting the choices of f_2 and g_2. This completes the proof of the claim. Now let $\mathcal{F}$ be a filterbase in $\mathrm{RO}(X_1)$ that converges to x. Since the collection of closed sets $\{\overline{\theta(V)}^{\beta X_2} : V \in \mathcal{F}\}$ has finite intersection property in the compact space βX_2,

$$\emptyset \neq \bigcap_{V\in\mathcal{F}} \overline{\theta(V)}^{\beta X_2} \subseteq \bigcap_{U\in\mathcal{N}_x} \overline{\theta(U)}^{\beta X_2} \subseteq X_2.$$

Hence $\bigcap_{V\in\mathcal{F}} \overline{\theta(V)} \neq \emptyset$. We have verified condition (2) in Theorem 3.1. Condition (3) in the same theorem follows by symmetry. Condition (1) in the theorem follows from Proposition 6.3. By Theorem 3.1, there is a homeomorphism φ associated with θ and thus with (R, h_1, h_2), as required. The final conclusion follows from Lemma 6.4. $\square$

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