Disjointness Preservers and Banach-Stone Theorems

Denny H. Leung and Wee Kee Tang

Abstract. Let Ω be a compact Hausdorff space. The space $C(\Omega)$ of continuous functions on Ω carries a number of structures. It is a Banach space (under the sup-norm), a vector lattice and a ring (under pointwise operations). The classical theorems of Banach-Stone, Kaplansky and Gelfand-Kolmogorov show that each of these structures on $C(\Omega)$ characterizes the space Ω up to homeomorphism. Within the last thirty years or so, a rich literature has been built up concerning mappings between function spaces that preserve the disjointness structure (biseparating maps or $\perp$-isomorphisms). These efforts have shown that in many cases, operators on function spaces that preserve various kinds of structures are $\perp$-isomorphisms. This lends a certain unity to various "preserver" results and highlights the utility of the concept of $\perp$-isomorphisms. In this paper, we will describe a general theory of $\perp$-isomorphisms and survey a number of applications, including applications to order (lattice) isomorphisms, ring and multiplicative isomorphisms, isometries and nonvanishing preservers.

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1. Introduction

There is a long and fruitful tradition of studying a mathematical object by means of looking at the space of mappings from it into a simple object of the same sort. For example, the dual group is a fundamental object in abstract harmonic analysis; likewise, the dual space of a locally convex topological vector space is part and parcel of the theory of such spaces. If Ω is a compact

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Hausdorff space, the space $C(\Omega)$ of continuous real valued functions on Ω is a natural "dual space" of Ω. (We will generally take the scalar field to be $\mathbb{R}$, although most of what will be discussed in the paper applies equally well to complex scalars.) Moreover, $C(\Omega)$ carries with it a wealth of structures. It is a Banach space under the norm $\|f\| = \sup_{\omega\in\Omega} |f(\omega)|$, a ring (with unit) under pointwise addition and multiplication and a vector lattice under pointwise order. Each of these aspects of $C(\Omega)$ has been shown to determine the space Ω up to homeomorphism. These are the famous classical theorems of Banach-Stone [10, 41], Gelfand-Kolmogorov [25] and Kaplansky [29]. The aim of this paper is to give a survey of some developments that arise out of these classical results, which we will refer to as theorems of Banach-Stone type. Particularly, since the 1990s, mappings that preserve "disjointness structures" – biseparating maps in the linear case, $\perp$-isomorphisms more generally – have been studied by many researchers. An important point that we would like to make is to promote the use of $\perp$-isomorphisms as a unifying concept in the study of Banach-Stone type theorems. A recent example of such a point of view is given in the paper [16]. For a general survey of Banach-Stone theorems up to around the year 2000, see [23].

Let us briefly summarize the contents of the paper. In §2, we recall the statements of the three classical theorems mentioned above. The definition of a biseparating map is given and it is shown that if $T : C(\Omega) \to C(\Sigma)$ is either an isometry, an algebra (ring) isomorphism or a vector lattice isomorphism, then it is biseparating. A detailed proof is given of the fact that a biseparating map $T : C(\Omega) \to C(\Sigma)$ induces a homeomorphism $\varphi : \Sigma \to \Omega$, with respect to which T can be represented as a weighted composition operator. (See Theorem 2.6.) Consequently, the three classical theorems can be unified under Theorem 2.6. §3 develops the theory of $\perp$-isomorphisms. Minimal assumptions are made on the sets of functions and the mappings involved. Even so, it is found that a $\perp$-isomorphism induces an isomorphism between the Boolean algebras of regular open sets between the underlying domain spaces (Theorem 3.3). Under further conditions, it is shown that the Boolean isomorphism gives rise to a homemorphism between the domain spaces. These can be viewed as "weak" Banach-Stone theorems. In §3.3, "strong" Banach-Stone theorems are given, that is, results where a $\perp$-isomorphism has a functional representation. Strong Banach-Stone theorems are seen to apply to a large class of function spaces. Finally, §4 contains applications of the results in §3 to a variety of settings. It is shown that in many cases lattice isomorphisms (Kaplansky's Theorem), ring isomorphisms (Gelfand-Kolmogorov Theorem), multiplicative isomorphisms (Milgram's Theorem), isometries (Banach-Stone Theorem) and nonvanishing preservers are $\perp$-isomophisms. Consequently, many results are consequences of, and can be extended by, characterization of $\perp$-isomorphisms.

2. Three Classical Theorems

Let Ω be a topological space. Denote by $C(\Omega)$ the vector space of all (real-valued) continuous functions on Ω. The space $C(\Omega)$ carries with it many structures. Indeed, it is an algebra under pointwise addition and multiplication. It is also a vector lattice under pointwise supremum and infimum. Finally, if Ω is compact Hausdorff, the space $C(\Omega)$ is a Banach space with the norm $\|f\| = \sup\{|f(\omega)| : \omega \in \Omega\}$. In the first half of the twentieth century, three remarkable theorems appeared that characterize the space Ω in terms of each of these structures of $C(\Omega)$.

Theorem 2.1 (Banach-Stone). *Let Ω, Σ be compact Hausdorff spaces and let $T : C(\Omega) \to C(\Sigma)$ be a linear isometry. There are a homeomorphism $\varphi : \Sigma \to \Omega$ and a function $h \in C(\Sigma)$ so that $|h(\sigma)| = 1$ for all $\sigma \in \Sigma$ and that*

$$Tf(\sigma) = h(\sigma)f(\varphi(\sigma)) \text{ for all } f \in C(\Omega) \text{ and all } \sigma \in \Sigma.$$

Theorem 2.1 was proved by Banach [10] for the case of compact metric spaces. The theorem was extended to compact Hausdorff spaces by Stone [41].

Theorem 2.2 (Gelfand-Kolmogorov). [25] *Let Ω, Σ be compact Hausdorff spaces and let $T : C(\Omega) \to C(\Sigma)$ be an algebra isomorphism. There is a homeomorphism $\varphi : \Sigma \to \Omega$ such that*

$$Tf(\sigma) = f(\varphi(\sigma)) \text{ for all } f \in C(\Omega) \text{ and all } \sigma \in \Sigma.$$

Theorem 2.3 (Kaplansky). [29] *Let Ω, Σ be compact Hausdorff spaces and let $T : C(\Omega) \to C(\Sigma)$ be a vector lattice isomorphism. There are a homeomorphism $\varphi : \Sigma \to \Omega$ and a function $h \in C(\Sigma)$ so that $h(\sigma) > 0$ for all $\sigma \in \Sigma$ and that*

$$Tf(\sigma) = h(\sigma)f(\varphi(\sigma)) \text{ for all } f \in C(\Omega) \text{ and all } \sigma \in \Sigma.$$

Theorem 2.3 is a special case of Kaplansky's result. For a discussion of the result in its full generality, see §4.1. In the intervening three quarters of a century, a large number of extensions and generalizations of these results have been obtained. A particularly fruitful concept that unifies the three classical theorems is that of disjointness preserving operators. Let Ω, Σ be topological spaces. Two functions $f, g \in C(\Omega)$, respectively, $C(\Sigma)$, are said to be *disjoint* if the pointwise product $fg = 0$. In terms of the lattice structure, f and g are disjoint if and only if $|f| \wedge |g| = 0$. Suppose that $A(\Omega)$ and $A(\Sigma)$ are vector subspaces of $C(\Omega)$ and $C(\Sigma)$ respectively. A linear operator $T : A(\Omega) \to A(\Sigma)$ is *disjointness preserving* if Tf, Tg are disjoint whenever f, g are disjoint functions in $A(\Omega)$. A *biseparating* operator is a linear bijection $T : A(\Omega) \to A(\Sigma)$ so that both T and T^{-1} are disjointness preserving. It is evident that if $A(\Omega)$ and $A(\Sigma)$ are algebras under pointwise operations, then every algebraic isomorphism $T : A(\Omega) \to A(\Sigma)$ is biseparating. A similar statement holds for lattice isomorphisms. Now we proceed to see that for compact Hausdorff spaces Ω and Σ, any linear isometry from $C(\Omega)$ onto $C(\Sigma)$ is biseparating. To do this, we make use of extreme points in the dual ball of $C(\Omega)$ and $C(\Sigma)$. Let C be a convex set in a vector space V. A point $x \in C$ is an *extreme point* of C if $x = \frac{1}{2}(y+z)$, $y, z \in C$, implies that $x = y = z$. Denote the set of extreme points of C by $\operatorname{ext} C$. It is easy to see that if V, W are vector spaces, $T : V \to W$ is a vector space isomorphism and x is an extreme point of $C \subseteq V$, then Tx is an extreme point of $T(C)$. The next result is due to Arens and Kelley, who used it in their proof of the Banach-Stone Theorem.

Proposition 2.4. [8] *Let Ω be a compact Hausdorff space and let $B_{C(\Omega)^*}$ be the closed ball of the dual space $C(\Omega)^*$. Then*

$$\operatorname{ext} B_{C(\Omega)^*} = \{\pm\delta_\omega : \omega \in \Omega\},$$

where δ_ω is the evaluation functional on $C(\Omega)$ given by $\delta_\omega(f) = f(\omega)$.

Proposition 2.5. *Let Ω and Σ be compact Hausdorff spaces. Every (onto) linear isometry $T : C(\Omega) \to C(\Sigma)$ is biseparating.*

Proof. Let f, g be disjoint functions in $C(\Omega)$ and let $\sigma \in \Sigma$. By Proposition 2.4, $\delta_\sigma \in \operatorname{ext} B_{C(\Sigma)^*}$. Since T^* is a vector space isomorphism and $T^*(B_{C(\Sigma)^*}) = B_{C(\Omega)^*}$, $T^*\delta_\sigma \in \operatorname{ext} B_{C(\Omega)^*}$. By Proposition 2.4, there exists $\omega \in \Omega$ and $\varepsilon = \pm 1$ so that $T^*\delta_\sigma = \varepsilon\delta_\omega$. Thus

$$Tf(\sigma) \cdot Tg(\sigma) = (T^*\delta_\sigma)(f) \cdot (T^*\delta_\sigma)(g) = \varepsilon^2\delta_\omega(f) \cdot \delta_\omega(g) = f(\omega) \cdot g(\omega) = 0.$$

This proves that Tf and Tg are disjoint. Hence T is disjointness preserving. The same applies to T^{-1} by symmetry. $\square$

The classical theorems of Banach-Stone, Gelfand-Kolmogorov and Kaplansky can now be unified and extended by the next result.

Theorem 2.6. [28] *Let Ω and Σ be compact Hausdorff spaces and let $T : C(\Omega) \to C(\Sigma)$ be a linear biseparating map. There are a homeomorphism $\varphi : \Sigma \to \Omega$ and a function $h \in C(\Sigma)$ so that $h(\sigma) \neq 0$ for all $\sigma \in \Sigma$ and that*

$$Tf(\sigma) = h(\sigma)f(\varphi(\sigma)) \text{ for all } f \in C(\Omega) \text{ and all } \sigma \in \Sigma.$$

In fact, Jarosz gave a description of general disjointness preserving linear maps $T : C(\Omega) \to C(\Sigma)$. As a result, he showed that every disjointness preserving linear bijection $T : C(\Omega) \to C(\Sigma)$ is biseparating and has the representation above. We will give a detailed proof of Theorem 2.6 that seems to us to be most amenable to generalization. For the remainder of the section, let Ω, Σ and T be as in Theorem 2.6. For any function $f \in C(\Omega)$, the *support* of f, $\operatorname{supp} f$, is the closure of the set $\{\omega : f(\omega) \neq 0\}$. Similarly for functions in $C(\Sigma)$.

Proposition 2.7. [5, Lemma 4] *If $f, g \in C(\Omega)$ and $\operatorname{supp} f \subseteq \operatorname{supp} g$, then $\operatorname{supp} Tf \subseteq \operatorname{supp} Tg$.*

Proof. Otherwise, there are $f, g \in C(\Omega)$ with $\operatorname{supp} f \subseteq \operatorname{supp} g$, yet $\operatorname{supp} Tf \not\subseteq \operatorname{supp} Tg$. Hence there exists $\sigma_0 \notin \operatorname{supp} Tg$ so that $Tf(\sigma_0) \neq 0$. Choose $h \in C(\Sigma)$ so that $h(\sigma_0) \neq 0$ and that h is disjoint from Tg. Since T^{-1} is disjointness preserving, $T^{-1}h$ and g are disjoint. Thus $T^{-1}h$ and f are disjoint. Therefore, h and Tf are disjoint, which contradicts the fact that h and Tf are both nonzero at σ_0. $\square$

For any $\sigma \in \Sigma$, set

$$\mathcal{F}_\sigma = \{\operatorname{supp} f : f \in C(\Omega), (Tf)(\sigma) \neq 0\}.$$

Lemma 2.8. *$\mathcal{F}_\sigma$ has the finite intersection property.*

Proof. Let $f_1, \ldots, f_m$ be functions in $C(\Omega)$ so that $Tf_i(\sigma) \neq 0$, $1 \leq i \leq m$. There exists a nonzero $g \in C(\Sigma)$ so that $\operatorname{supp} g \subseteq \operatorname{supp} Tf_i$, $1 \leq i \leq m$. Apply Proposition 2.7 to T^{-1} to see that $\operatorname{supp} T^{-1}g \subseteq \operatorname{supp} f_i$, $1 \leq i \leq m$. Since T is a bijection and $g \neq 0$, $T^{-1}g \neq 0$. Thus $\operatorname{supp} T^{-1}g$ is a nonempty set contained in $\bigcap_{i=1}^m \operatorname{supp} f_i$. $\square$

Lemma 2.9. *For any $\sigma \in \Sigma$, $\bigcap \mathcal{F}_\sigma$ contains exactly one point in Ω.*

Proof. Obviously, $\mathcal{F}_\sigma$ consists of closed sets in the compact Hausdorff space Ω. It follows from Lemma 2.8 that $\bigcap \mathcal{F}_\sigma$ is nonempty. Suppose, if possible, that ω_1, ω_2 are two distinct points in $\bigcap \mathcal{F}_\sigma$. Choose a pair of disjoint functions $h_1, h_2 \in C(\Omega)$ so that $h_i = 1$ on a neighborhood of ω_i, $i = 1, 2$. Let $f \in C(\Omega)$ be chosen so that $Tf(\sigma) \neq 0$. By definition, $\omega_i \in \operatorname{supp} f$, $i = 1, 2$. Since $h_1 f$ and $h_2 f$ are disjoint and T is disjointness preserving, $T(h_1 f)$ and $T(h_2 f)$ are disjoint. Without loss of generality, we may assume that $T(h_1 f)(\sigma) = 0$. Then $T((1 - h_1)f)(\sigma) = Tf(\sigma) \neq 0$. Since $\omega_1 \in \mathcal{F}_\sigma$, this would imply that $\omega_1 \in \operatorname{supp}(1 - h_1)f$, which is clearly false by choice of h_1. This completes the proof of the lemma. $\square$

Define $\varphi : \Sigma \to \Omega$ by setting $\{\varphi(\sigma)\} = \bigcap \mathcal{F}_\sigma$. By symmetry, we may define

$$\mathcal{F}_\omega = \{\operatorname{supp} Tf : f \in C(\Omega), f(\omega) \neq 0\}$$

for any $\omega \in \Omega$. Then there is a well-defined function $\psi : \Omega \to \Sigma$ so that $\{\psi(\omega)\} = \bigcap \mathcal{F}_\omega$ for all $\omega \in \Omega$.

Lemma 2.10. *$\varphi : \Sigma \to \Omega$ is a homeomorphism with inverse ψ.*

Proof. We will show that $\psi(\varphi(\sigma)) = \sigma$ for all $\sigma \in \Sigma$ and that φ is continuous. The lemma then follows by symmetry.

Suppose that $\sigma \in \Sigma$ and $\omega = \varphi(\sigma)$. Assume, if possible, that $\sigma' = \psi(\omega) \neq \sigma$. Let $f \in C(\Omega)$ be such that $Tf(\sigma) \neq 0$ and that $\sigma' \notin \operatorname{supp} Tf$. There exists $h \in C(\Sigma)$ disjoint from Tf so that $h = 1$ on a neighborhood of σ'. Choose $g \in C(\Omega)$ so that $g(\omega) \neq 0$. Since $h \cdot Tg$ and Tf are disjoint, so are $T^{-1}(h \cdot Tg)$ and f. As $Tf(\sigma) \neq 0$, $\omega \in \operatorname{supp} f$. Hence $T^{-1}(h \cdot Tg)(\omega) = 0$. Therefore,

$$0 \neq g(\omega) = T^{-1}(h \cdot Tg)(\omega) + T^{-1}((1 - h) \cdot Tg)(\omega) = T^{-1}((1 - h) \cdot Tg)(\omega).$$

It follows that $\sigma' \in \operatorname{supp}(1-h) \cdot Tg$, contrary to the choice of h. This completes the proof that $\psi(\varphi(\sigma)) = \sigma$.

If φ is not continuous, then making use of compactness of Ω, there is a net $(\sigma_\alpha)_\alpha$ in Σ converging to some σ_0 so that $(\varphi(\sigma_\alpha))_\alpha$ converges to

$\omega' \neq \omega_0 := \varphi(\sigma_0)$. Let $f \in C(\Omega)$ be such that $Tf(\sigma_0) \neq 0$. There exists α_0 so that $Tf(\sigma_\alpha) \neq 0$ for all $\alpha \succeq \alpha_0$. By definition of φ, $\varphi(\sigma_\alpha) \in \operatorname{supp} f$ for all $\alpha \succeq \alpha_0$. Thus $\omega' \in \operatorname{supp} f$. But this shows that $\omega' \in \bigcap \mathcal{F}_{\sigma_0}$ and hence $\omega' = \omega$, contrary to the assumption. □

Proof of Theorem 2.6. We will show that if $f(\varphi(\sigma)) = 0$, then $Tf(\sigma) = 0$. Once this is shown, define $h = T1$. For any $f \in C(\Omega)$ and any $\sigma \in \Sigma$, $f - f(\varphi(\sigma))1$ vanishes at $\varphi(\sigma)$. Hence

$$0 = T[f - f(\varphi(\sigma))1](\sigma) = Tf(\sigma) - f(\varphi(\sigma))h(\sigma).$$

Thus $Tf(\sigma) = h(\sigma)f(\varphi(\sigma))$, as claimed. Furthermore, since T is a surjection, $h(\sigma) \neq 0$ for any $\sigma \in \Sigma$.

Suppose that, contrary to the claim above, there are $f \in C(\Omega)$ and $\sigma \in \Sigma$ so that $f(\varphi(\sigma)) = 0$ yet $Tf(\sigma) \neq 0$. By definition of φ, $\omega := \varphi(\sigma) \in \operatorname{supp} f$. Thus $\omega \in \overline{|f|^{-1}(0,r)}$ for any $r > 0$. Define

$$U_n = |f|^{-1}(\frac{1}{(3n+5)^2}, \frac{1}{(3n+1)^2}),\ n \in \mathbb{N}.$$

Then $\omega \in \overline{\bigcup_n U_n} = \overline{\bigcup_n U_{2n-1}} \cup \overline{\bigcup_n U_{2n}}$. Without loss of generality, assume that $\omega \in \overline{\bigcup_n U_{2n-1}}$. Set

$$V_n = |f|^{-1}(\frac{1}{(6n+3)^2}, \frac{1}{(6n-3)^2}),\ n \in \mathbb{N}.$$

Then (V_n) is a sequence of disjoint open sets so that $\overline{U_{2n-1}} \subseteq V_n$ for all n. For each n, choose a function $h_n \in C(\Omega)$ so that $0 \leq h_n \leq 1$, $h_n = 1$ on U_{2n-1} and $h_n = 0$ outside V_n. The sequence of functions $(nh_nf)_n$ is pairwise disjoint and $\|nh_nf\| \leq \frac{n}{(6n-3)^2} \to 0$. Hence the sum $g := \sum nh_nf$ converges in $C(\Omega)$. For each n, $g - nf = 0$ on the set U_{2n-1}. By definition of φ, this implies that $T(g - nf)(\sigma') = 0$ for all $\sigma' \in \varphi^{-1}(U_{2n-1})$. Choose a net (ω_α) in $\bigcup_n U_{2n-1}$ that converges to ω. Let $n_\alpha \in \mathbb{N}$ be such that $\omega_\alpha \in U_{2n_\alpha-1}$. Set $\sigma_\alpha = \varphi^{-1}(\omega_\alpha)$. Since $f(\omega) = 0$, $\omega \notin \overline{U_n}$ for any n. Thus $\lim_\alpha n_\alpha = \infty$. Note that (σ_α) converges to σ. Therefore,

$$Tf(\sigma) = \lim_\alpha Tf(\sigma_\alpha) = \lim_\alpha \frac{1}{n_\alpha} Tg(\sigma_\alpha) = 0,$$

contrary to the choices of f and σ. □

3. Isomorphism of disjointness structure

Results in §2 may serve to convince the reader that biseparating maps are worthy of study in their own right. Indeed, plenty of results concerning biseparating maps have been obtained in the past thirty years or so. Most of these are in the context of linear or at least additive maps. Since surjective additive maps between vector spaces are linear maps over the field of rational numbers, the results remain mainly "linear" in character. Very recently, several papers [16, 18, 19] appeared that took the study of isomorphisms of disjointness structure, or $\perp$-isomorphisms, to very general settings. It is shown that even for function spaces with minimal structure, analysis of $\perp$-isomorphisms can still bear fruitful results. The aim of this section is to describe this general approach to $\perp$-isomorphisms. Earlier results on biseparating maps, in spaces of (vector-valued) continuous functions, uniformly continuous functions, Lipschitz functions and differentiable functions, will be seen as consequences. Applications to theorems of Banach-Stone type will be considered in the next section.

3.1. $\perp$-isomorphisms

Let Ω, X be Hausdorff topological spaces and let $A(\Omega, X)$ be a subset of $C(\Omega, X)$, the set of continuous functions $f : \Omega \to X$. For $f, g \in A(\Omega, X)$, let

$$[f \neq g] = \{\omega \in \Omega : f(\omega) \neq g(\omega)\},\ \operatorname{supp}_g f = \overline{[f \neq g]} \text{ and } \sigma_g(f) = \operatorname{int} \operatorname{supp}_g f.$$

Following [16], we define the following relations for $f, g, h \in A(\Omega, X)$.

1. $f \perp_h g$: $[f \neq h] \cap [g \neq h] = \emptyset$.
2. $f \subseteq_h g$: $\sigma_h(f) \subseteq \sigma_h(g)$.

The definitions of $\perp_h$ and $\subseteq_h$ may appear asymmetrical as one uses sets of the form $[f \neq h]$ while the other uses $\sigma_h(f)$. However, it is easy to see that $f \perp_h g$ if and only if $\sigma_h(f) \cap \sigma_h(g) = \emptyset$. Similarly, let $A(\Sigma, Y)$ be a subset of $C(\Sigma, Y)$, where Σ, Y are Hausdorff topological spaces. Assume that $T : A(\Omega, X) \to A(\Sigma, Y)$ is a bijection. Given $h \in A(\Omega, X)$, say that T is a $\perp_h$- *isomorphism* if

$$f \perp_h g \iff Tf \perp_{Th} Tg \text{ for all } f, g \in A(\Omega, X).$$

$\subseteq_h$-*isomorphism* is defined similarly. Clearly, a biseparating map in the sense of §2 is precisely a $\perp_0$ isomorphism, provided $T0 = 0$. $\perp_h$-isomorphism is a generalization of biseparating map to the nonlinear context. The set $A(\Omega, X)$ is said to be *h-weakly regular* for some $h \in A(\Omega, X)$ if

$$\Sigma_h = \{\sigma_h(f) : f \in A(\Omega, X)\} \text{ is a basis for the topology on } X.$$

Weak regularity is a basic assumption to ensure that there are sufficient functions in $A(\Omega, X)$ and $A(\Sigma, Y)$ to yield a nontrivial theory. The following simple yet important result is noted and used in [16]. Its ancestry can be traced back to at least [5, Lemma 4].

Proposition 3.1. *Let $T : A(\Omega, X) \to A(\Sigma, Y)$ be a bijection, where $A(\Omega, X)$ and $A(\Sigma, Y)$ are h- and Th-weakly regular respectively. Then T is a $\perp_h$-isomorphism if and only if it is a $\subseteq_h$-isomorphism.*

A set U in Ω is a *regular open set* if $U = \operatorname{int} \overline{U}$. All sets of the form $\sigma_h(f)$ are regular open sets. Denote the collection of all regular open sets in Ω by $\mathrm{RO}(\Omega)$. $\mathrm{RO}(\Omega)$ is a Boolean algebra with $0 = \emptyset$, $1 = \Omega$, lattice operations $U \wedge V = U \cap V$, $U \vee V = \operatorname{int} \overline{U \cup V}$ and negation $\neg U = \operatorname{int}(\Omega \backslash U)$. See [41]. If Ω is a regular topological space, then $\mathrm{RO}(\Omega)$ is a basis for the topology on Ω.

Let $T : A(\Omega, X) \to A(\Sigma, Y)$ be a $\subseteq_h$-isomorphism, where $A(\Omega, X)$ and $A(\Sigma, Y)$ are h- and Th-weakly regular respectively. Define a map θ_h : $\mathrm{RO}(\Omega) \to \mathrm{RO}(\Sigma)$ by

$$\theta_h(U) = \operatorname{int}\overline{\bigcup\{\sigma_{Th}(Tf) : f \in A(\Omega, X), \sigma_h(f) \subseteq U\}}. \tag{3.1}$$

It can be shown that θ_h is a Boolean isomorphism from $\mathrm{RO}(\Omega)$ onto $\mathrm{RO}(\Sigma)$. In fact, its inverse is $\theta_{Th} : \mathrm{RO}(\Sigma) \to \mathrm{RO}(\Omega)$. Furthermore, if $f \in A(\Omega, X)$ and $U \in \mathrm{RO}(\Omega)$, then $f = h$ on U if and only if $Tf = Th$ on $\theta_h(U)$. In fact, we obtain a fundamental characterization of $\perp_h$-isomorphisms.

Theorem 3.2. *Let $T : A(\Omega, X) \to A(\Sigma, Y)$ be a bijection, where $A(\Omega, X)$ and $A(\Sigma, Y)$ are h- and Th-weakly regular respectively. Then T is a $\perp_h$-isomorphism if and only if there is a Boolean isomorphism $\theta_h : \mathrm{RO}(\Omega) \to \mathrm{RO}(\Sigma)$ so that for any $f \in A(\Omega, X)$ and $U \in \mathrm{RO}(X)$, $f = h$ on U if and only if $Tf = Th$ on $\theta_h(U)$.*

In Theorem 3.2, we say that θ_h is *associated with* (T, h). In general, if T is a $\perp_h$ isomorphism for different h's, the associated Boolean isomorphisms θ_h may well depend on h. Some way of "linking" different functions in $A(\Omega, X)$ and $A(\Sigma, Y)$ is needed in order to "uniformize" the θ_h's.

Call a set $A(\Omega, X) \subseteq C(\Omega, X)$ *weakly regular* if $A(\Omega, X)$ is h-weakly regular for all $h \in A(\Omega, X)$. Suppose that $T : A(\Omega, X) \to A(\Sigma, Y)$ is a bijection between weakly regular sets of functions that is a $\perp$*-isomorphism*, i.e., T is a $\perp_h$-isomorphism for all $h \in A(\Omega, X)$. Consider the following "linking" condition.

(L) If $h_1, h_2 \in A(\Omega, X)$, $U \in \mathrm{RO}(\Omega)$ and $\omega \notin \overline{U}$, then there exist $f \in A(\Omega, X)$ and $V \in \mathrm{RO}(\Omega)$ containing ω so that

$$f = \begin{cases} h_1 & \text{on } U \\ h_2 & \text{on } V. \end{cases}$$

A set of functions $A(\Omega, X)$ is *nowhere trivial* if for any $\omega \in \Omega$, there are $h_1, h_2 \in A(\Omega, X)$ so that $h_1(\omega) \neq h_2(\omega)$. If Ω is a regular topological space and $A(\Omega, X)$ is nowhere trivial and satisfies condition (L), then $A(\Omega, X)$ is weakly regular.

Theorem 3.3. *Let Ω, Σ be regular topological spaces. Assume that $A(\Omega, X)$ and $A(\Sigma, Y)$ are nowhere trivial and satisfy condition (L). A bijection $T : A(\Omega, X) \to A(\Sigma, Y)$ is a $\perp$-isomorphism if and only if there is Boolean isomorphism $\theta : \mathrm{RO}(\Omega) \to \mathrm{RO}(\Sigma)$ so that for all $f, g \in A(\Omega, X)$ and all $U \in \mathrm{RO}(\Omega)$, $f = g$ on U if and only if $Tf = Tg$ on $\theta(U)$.*

In order to prove Theorem 3.3, we first require a lemma.

Lemma 3.4. *Assume that $h_1, h_2 \in A(\Omega, X)$, $U \in \mathrm{RO}(\Omega)$ so that $h_1 = h_2$ on U. Then $\theta_{h_1}(U) = \theta_{h_2}(U)$.*

Proof. Assume that $\theta_{h_2}(U) \not\subseteq \overline{\theta_{h_1}(U)}$. Since T is a bijection and $A(\Sigma, Y)$ is weakly regular, there exists $f \in A(\Omega, X)$ such that

$$\emptyset \neq \sigma_{Th_2}(Tf) \subseteq \theta_{h_2}(U)\backslash\overline{\theta_{h_1}(U)}.$$

By (3.1) and the fact that T is a $\subseteq$-isomorphism, $\theta_{h_2}(\sigma_{h_2}(f)) = \sigma_{Th_2}(Tf)$. Then $\emptyset \neq \sigma_{h_2}(f) = \theta_{h_2}^{-1}(\sigma_{Th_2}(Tf)) \subseteq U$. Hence $\sigma_{h_2}(f)$ is a nonempty set disjoint from $\overline{\neg U}$. By condition (L), there exist $g \in A(\Omega, X)$ and a nonempty set $V \in \mathrm{RO}(\Omega)$, $V \subseteq \sigma_{h_2}(f)$, so that

$$g = \begin{cases} h_1 & \text{on } \neg U, \\ f & \text{on } V. \end{cases}$$

Now

1. $\theta_f(V) \subseteq \theta_f(\sigma_{h_2}(f)) = \theta_f(\sigma_f(h_2)) = \sigma_{Tf}(Th_2) = \sigma_{Th_2}(Tf)$.
2. $Tg = Th_1$ on $\theta_{h_1}(\neg U) = \neg\theta_{h_1}(U) = \mathrm{int}[\theta_{h_1}(U)^c] \supseteq \sigma_{Th_2}(Tf)$.
3. $Tg = Tf$ on $\theta_f(V)$.
4. $Th_1 = Th_2$ on $\theta_{h_2}(U) \supseteq \sigma_{Th_2}(Tf)$.

Here, we have applied Theorem 3.2 for items 2-4. It follows that $Tf = Th_2$ on $\theta_f(V)$. However, $\theta_f(V)$ is a nonempty open subset of $\sigma_{Th_2}(Tf) = \mathrm{int}\,\overline{[Tf \neq Th_2]}$. So we have reached a contradiction. Therefore, $\theta_{h_2}(U) \subseteq \overline{\theta_{h_1}(U)}$. Since $\theta_{h_1}(U)$ is a regular open set, $\theta_{h_2}(U) \subseteq \theta_{h_1}(U)$. The lemma follows by symmetry. □

Proof of Theorem 3.3. Taking into account Theorem 3.2, it suffices to show that $\theta_{h_1} = \theta_{h_2}$ for any $h_1, h_2 \in A(\Omega, X)$. Suppose that there exists $U \in \mathrm{RO}(\Omega)$ so that $\theta_{h_2}(U) \not\subseteq \theta_{h_1}(U)$, so that in fact $\theta_{h_2}(U) \not\subseteq \overline{\theta_{h_1}(U)}$. By condition (L) for $A(\Sigma, Y)$, there exists $g \in A(\Sigma, Y)$ and a nonempty set $V \in \mathrm{RO}(\Sigma)$, $V \subseteq \theta_{h_2}(U)\backslash\overline{\theta_{h_1}(U)}$, so that

$$g = \begin{cases} Th_1 & \text{on } \theta_{h_1}(U), \\ Th_2 & \text{on } V. \end{cases}$$

Apply Lemma 3.4 on $A(\Sigma, Y)$. We find that

$$\theta_g(\theta_{h_1}(U)) = \theta_{Th_1}(\theta_{h_1}(U)) = U \quad \text{and} \quad \theta_g(V) = \theta_{Th_2}(V) = \theta_{h_2}^{-1}(V).$$

Since $\theta_{h_1}(U) \cap V = \emptyset$ and θ_g is a Boolean isomorphism,

$$U \cap \theta_{h_2}^{-1}(V) = \theta_g(\theta_{h_1}(V)) \cap \theta_g(V) = \emptyset.$$

However, $\theta_{h_2}^{-1}(V)$ is a nonempty subset of $\theta_{h_2}^{-1}(\theta_{h_2}(U)) = U$. The contradiction shows that $\theta_{h_2}(U) \subseteq \theta_{h_1}(U)$. The reverse inclusion follows by symmetry. □

In Theorem 3.3, say that θ is *associated with* T. We list a few examples of sets of functions satisfying condition (L). Another example is given in Lemma 4.2 below.

Example. (a) Let Ω be a completely regular Hausdorff space and let X be a convex set in a Hausdorff topological vector space. The space $C(\Omega, X)$ consists of all continuous functions from Ω into X.

(b) Let Ω be a metric space and let X be a convex set in a normed space. Denote by $U(\Omega, X)$, $U_*(\Omega, X)$, $\mathrm{Lip}(\Omega, X)$, $\mathrm{Lip}_*(\Omega, X)$, respectively, the set of uniformly continuous functions, the set of bounded uniformly continuous functions, the set of Lipschitz functions and the set of bounded Lipschitz functions from Ω to X.

(c) Let Ω be an open set in a Banach space $\mathcal{Z}$ and let $\mathcal{X}$ be a Banach space. For $p \in \mathbb{N} \cup \{\infty\}$, denote by $C^p(\Omega, \mathcal{X})$ the space of all p-times continuously (Fréchet) differentiable $\mathcal{X}$-valued functions on Ω. To ensure that there are "sufficiently many" functions in $C^p(\Omega, \mathcal{X})$, we assume that there exists a bump function in $C^p(\mathcal{Z})$, i.e., a function $\xi \in C^p(\mathcal{Z})$ that has nonempty bounded support in $\mathcal{Z}$.

To see that all of the spaces $A(\Omega, X)$ above satisfy condition (L), first observe that if $\omega_0 \in \Omega$, $U \in \mathrm{RO}(\Omega)$ and $\omega_0 \notin \overline{U}$, then there exist $\xi : \Omega \to [0, 1]$, $V \in \mathrm{RO}(X)$ containing ω_0 so that $\xi = 0$ on V and $\xi = 1$ on U. Moreover, for the situation in (b), we can choose ξ to be Lipschitz, and for case (c), we can choose $\xi \in C^p(\Omega)$. Given $h_1, h_2 \in A(\Omega, X)$, it is easy to verify that $f(\omega) = \xi(\omega)h_1(\omega) + (1 - \xi(\omega))h_2(\omega)$ defines a function in $A(\Omega, X)$ that is equal to h_1 on U and h_2 on V.

Remark 3.5. Condition (L) is a condition on $A(\Omega, X)$, respectively, $A(\Sigma, Y)$, that guarantees that the Boolean isomorphisms θ_h are independent of h. Alternatively, we may impose conditions on T to warrant the same outcome. For example, if X, Y are Hausdorff topological groups, then $C(\Omega, X)$ is a topological group under pointwise group operations. Suppose that $A(\Omega, X), A(\Sigma, Y)$ are subgroups of $C(\Omega, X), C(\Sigma, Y)$ respectively and $T : A(\Omega, X) \to A(\Sigma, Y)$ is a group isomorphism as well as a $\perp_h$ isomorphism for some $h \in A(\Omega, X)$. Then routine verification shows that T is a $\perp$-isomorphism and for any $k \in A(\Omega, X)$, $\theta_h(U) = \theta_k(U)$ for all $U \in \mathrm{RO}(X)$. In particular, the situation occurs if X and Y are Hausdorff topological vector spaces, $A(\Omega, X), A(\Sigma, Y)$ are respective subspaces of $C(\Omega, X), C(\Sigma, Y)$, and $T : A(\Omega, X) \to A(\Sigma, Y)$ is an additive $\perp_0$-isomorphism.

3.2. Homeomorphism associated with a $\perp$-isomorphism

Theorem 3.3 allows us to associate a Boolean isomorphism with a $\perp$-isomorphism. Unfortunately, in general, a Boolean isomorphism $\theta : \mathrm{RO}(\Omega) \to \mathrm{RO}(\Sigma)$ need not induce a homeomorphism $\varphi : \Omega \to \Sigma$.

Example. [16] Let Ω be a topological space and let Σ be a dense open set in Ω. Then the map $\theta_\Sigma : \mathrm{RO}(\Omega) \to \mathrm{RO}(\Sigma)$ given by $\theta_\Sigma(U) = U \cap \Sigma$ is a Boolean isomorphism. In particular, let S^1 be the unit circle in the complex plane. The sets $(0, 1)$ and $S^1\backslash\{1\}$ are homeomorphic and open and dense in $[0, 1]$ and S^1 respectively. Hence we have a chain of Boolean isomorphisms

$$\mathrm{RO}([0, 1]) \leftrightarrow \mathrm{RO}((0, 1)) \leftrightarrow \mathrm{RO}(S^1\backslash\{1\}) \leftrightarrow \mathrm{RO}(S^1).$$

But of course $[0, 1]$ and S^1 are not homeomorphic.

The next result characterizes the Boolean isomorphisms that induce homeomorphisms. If $\omega \in \Omega$, where Ω is a topological space, let $\mathcal{N}_\omega$ be the family of open neighborhoods of ω.

Proposition 3.6. *Let Ω, Σ be Hausdorff topological spaces and let $\theta : \mathrm{RO}(\Omega) \to \mathrm{RO}(\Sigma)$ be a Boolean isomorphism. Assume that*

1. *For any $\omega \in \Omega$, there exists $\sigma \in \Sigma$ such that for any $V \in \mathcal{N}_\sigma$, there exists $U \in \mathrm{RO}(\Omega)$ containing ω such that $\theta(U) \subseteq V$.*
2. *For any $\sigma \in \Sigma$, there exists $\omega \in \Omega$ such that for any $U \in \mathcal{N}_\omega$, there exists $V \in \mathrm{RO}(\Sigma)$ containing σ such that $\theta^{-1}(V) \subseteq U$.*

Then there exists a homeomorphism $\psi : \Omega \to \Sigma$ such that $\psi(U) = \theta(U)$ for any $U \in \mathrm{RO}(X)$. Conversely, if Ω, Σ are regular topological spaces and there is a homeomorphism ψ such that $\psi(U) = \theta(U)$ for any $U \in \mathrm{RO}(X)$, then conditions 1 and 2 hold.

Given conditions 1 and 2, define $\psi(\omega) = \sigma$ when ω and σ are related by condition 1. Similarly, define $\varphi(\sigma) = \omega$ when σ and ω are related by condition 2. One can check that $\psi : \Omega \to \Sigma$ and $\varphi : \Sigma \to \Omega$ are continuous functions that are mutual inverses. Proposition 3.6 can be applied to obtain general versions of Theorem 2.6. A homeomorphism $\psi : \Omega \to \Sigma$ is *associated with* T if for any $U \in \mathrm{RO}(\Omega)$ and any $f, g \in A(\Omega, X)$, $f = g$ on U if and only if $Tf = Tg$ on $\psi(U)$.

Theorem 3.7. *Suppose that $A(\Omega, X), A(\Sigma, Y)$ are nowhere trivial subsets of $C(\Omega, X)$ and $C(\Sigma, Y)$ respectively that satisfy condition (L), where X, Y are Hausdorff spaces and Ω, Σ are compact Hausdorff. If $T : A(\Omega, X) \to A(\Sigma, Y)$ is a $\perp$-isomorphism, then there is a homeomorphism $\psi : \Omega \to \Sigma$ associated with T.*

Proof. By Theorem 3.3, there is a Boolean isomorphism $\theta : \mathrm{RO}(\Omega) \to \mathrm{RO}(\Sigma)$ associated with T. Let us verify condition 1 in Proposition 3.6. Condition 2 follows by symmetry. Fix $\omega \in \Omega$. By assumption, there are functions $h_1, h_2 \in A(\Omega, X)$ so that $h_1(\omega) \neq h_2(\omega)$. The family $\{\overline{\theta(U)} : \omega \in U \in \mathrm{RO}(\Omega)\}$ has the finite intersection property and hence has nonempty intersection. Suppose that there are two distinct points $\sigma_1, \sigma_2 \in \bigcap\{\overline{\theta(U)} : \omega \in U \in \mathrm{RO}(\Omega)\}$. By condition (L), there are $V_1, V_2 \in \mathrm{RO}(\Sigma)$ and $f \in A(\Sigma, Y)$ so that $\sigma_i \in V_i$ and $f = Th_i$ on V_i, $i = 1, 2$. Thus $T^{-1}f = h_i$ on $\theta^{-1}(V_i)$. However, if $\omega \in U \in \mathrm{RO}(\Omega)$, then $\theta(U) \cap V_i \neq \emptyset$ and hence $U \cap \theta^{-1}(V_i) \neq \emptyset$. It follows that $\omega \in \overline{\theta^{-1}(V_1)} \cap \overline{\theta^{-1}(V_2)}$. By continuity of $T^{-1}f$, this would mean that $h_1(\omega) = T^{-1}f(\omega) = h_2(\omega)$, which is a contradiction. Therefore, the intersection $\bigcap\{\overline{\theta(U)} : \omega \in U \in \mathrm{RO}(\Omega)\}$ contains a unique point σ.

If condition 1 of Proposition 3.6 fails, there exists $V \in \mathcal{N}_\sigma$ such that $\overline{\theta(U)} \cap V^c \neq \emptyset$ for all $U \in \mathrm{RO}(\Omega) \cap \mathcal{N}_\omega$. Using compactness again, there exists σ' such that $\sigma' \in \overline{\theta(U)} \cap V^c$ for all $U \in \mathrm{RO}(\Omega) \cap \mathcal{N}_\omega$. Clearly, $\sigma' \neq \sigma$ and both belong to the intersection of the family $\{\overline{\theta(U)} : \omega \in U \in \mathrm{RO}(\Omega)\}$, contrary to the previous paragraph. □

Theorem 3.7 extends to the case of complete metric domains, provided the sets of functions satisfy an additional linking condition. Let Ω be a complete metric space, X be a Hausdorff topological space and let $A(\Omega, X)$ be a subset of $C(\Omega, X)$. A sequence (ω_n) in Ω is *separated* if $\inf_{m\neq n} d(\omega_m, \omega_n) > 0$.

(L_s) Let $h_1, h_2 \in A(\Omega, X)$ and let (ω_n) be a separated sequence in Ω. Then there exists $f \in A(\Omega, X)$ and $U_1, U_2 \in \mathrm{RO}(\Omega)$ so that each U_i contains infinitely many ω_n and that $f = h_i$ on U_i, $i = 1, 2$.

Theorem 3.8. *Suppose that $A(\Omega, X), A(\Sigma, Y)$ are nowhere trivial subsets of $C(\Omega, X)$ and $C(\Sigma, Y)$ respectively that satisfy conditions (L) and (L_s), where X, Y are Hausdorff spaces and Ω, Σ are complete metric spaces. If $T : A(\Omega, X) \to A(\Sigma, Y)$ is a $\perp$-isomorphism, then there is a homeomorphism $\psi : \Omega \to \Sigma$ associated with T.*

Sketch of proof. By Theorem 3.3, there is a Boolean isomorphism $\theta : \mathrm{RO}(\Omega) \to \mathrm{RO}(\Sigma)$ associated with T. A bit of reflection shows that in order to verify condition 1 of Proposition 3.6, it is suffices to prove that if $\omega \in U_n \in \mathrm{RO}(\Omega)$, $\operatorname{diam} U_n \to 0$ and $\sigma_n \in \theta(U_n)$ for all n, then (σ_n) converges in Σ. Fix functions $h_1, h_2 \in A(\Omega, X)$ so that $h_1(\omega) \neq h_2(\omega)$. If (σ_n) fails to be convergent, then either the sequence has no accumulation point, or at least two accumulation points. In either case, from condition (L) or (L_s), there are $f \in A(\Sigma, Y)$ and $V_1, V_2 \in \mathrm{RO}(\Sigma)$ so that V_i contain infinitely many σ_n and $f = Th_i$ on V_i, $i = 1, 2$. As in the proof of Theorem 3.7, $\omega \in \overline{\theta^{-1}(V_i)}$, $i = 1, 2$ and $T^{-1}f = h_i$ on $\overline{\theta^{-1}(V_i)}$, which leads to a contradiction. □

3.3. Representation

In many cases, it is possible to improve Theorems 3.7 and 3.8 by giving a functional representation of the $\perp$-isomorphism T.

Proposition 3.9. *Let Ω, Σ, X, Y be Hausdorff spaces. Suppose that $T : A(\Omega, X) \to A(\Sigma, Y)$ is a bijection, where $A(\Omega, X), A(\Sigma, Y)$ are subsets of $C(\Omega, X)$ and $C(\Sigma, Y)$ respectively. Assume that there is a homeomorphism $\psi : \Omega \to \Sigma$ that is associated with T. If $f, g \in A(\Omega, X)$, $\omega_0 \in \Omega$ and there exists $h \in A(\Omega, X)$ so that $\omega_0 \in \overline{\operatorname{int}[h = f]} \cap \overline{\operatorname{int}[h = g]}$, then $Tf(\psi(\omega_0)) = Tg(\psi(\omega_0))$.*

Proof. Let $U = \operatorname{int}[h = f]$ and $V = \operatorname{int}[h = g]$. By assumption, $Th = Tf$ on $\psi(U)$ and $Th = Tf$ on $\psi(V)$. Since ψ is a homeomorphism and $\omega_0 \in \overline{U} \cap \overline{V}$, $\psi(\omega_0) \in \overline{\psi(U)} \cap \overline{\psi(V)}$. By continuity of Tf, Tg and Th, $Tf(\psi(\omega_0)) = Th(\psi(\omega_0)) = Tg(\psi(\omega_0))$. □

From the example following Theorem 3.3, the spaces listed there all satisfy condition (L).

Theorem 3.10. *Let Ω, Σ be a first countable compact Hausdorff topological space and let X, Y be a convex sets in a Hausdorff topological vector spaces, with X, Y containing more than one point. If $T : C(\Omega, X) \to C(\Sigma, Y)$ is a $\perp$-isomorphism, then there are a homeomorphism $\psi : \Omega \to \Sigma$ and a function $\Phi : \Omega \times X \to Y$ so that*

$$Tf(\psi(\omega)) = \Phi(\omega, f(\omega)) \text{ for all } f \in C(\Omega, X) \text{ and all } \omega \in \Omega.$$

Sketch of proof. As mentioned above, $C(\Omega, X)$ and $C(\Omega, Y)$ both satisfy condition (L). Hence there is a homeomorphism $\psi : \Omega \to \Sigma$ associated with T by Theorem 3.7. For any $x \in X$, let $g_x \in C(\Omega, X)$ be the constant function with value x. Define $\Phi : \Omega \times X \to Y$ by $\Phi(\omega, x) = (Tg_x)(\psi(\omega))$. Let $\omega_0 \in \Omega$ and $f \in C(\Omega, X)$. Set $x = f(\omega_0)$. Using the first countability of Ω, one can easily construct $h \in C(\Omega, X)$ so that $\omega_0 \in \overline{\operatorname{int}[h = f]} \cap \overline{\operatorname{int}[h = g_x]}$. By Proposition 3.9,

$$Tf(\psi(\omega_0)) = (Tg_x)(\psi(\omega_0)) = \Phi(\omega_0, x) = \Phi(\omega_0, f(\omega_0)).$$

This completes the proof of the theorem. □

It is not hard to see that in this case Φ is a continuous function on $\Omega \times X$. In [19], it is shown that if $\mathcal{X}$ is a Banach space, then the spaces in the example (taking $X = \mathcal{X}$ where appropriate) satisfy condition (L_s). Hence we obtain the next result similarly.

Theorem 3.11. *Let $A(\Omega, \mathcal{X})$ be one of the spaces $U(\Omega, \mathcal{X}), U_*(\Omega, \mathcal{X}), \mathrm{Lip}(\Omega, \mathcal{X})$ or $\mathrm{Lip}_*(\Omega, \mathcal{X})$, where Ω is a complete metric space and $\mathcal{X}$ is a Banach space. Similarly for $A(\Sigma, \mathcal{Y})$. If $T : A(\Omega, \mathcal{X}) \to A(\Sigma, \mathcal{Y})$ is a $\perp$-isomorphism, then there are a homeomorphism $\psi : \Omega \to \Sigma$ and a function $\Phi : \Omega \times \mathcal{X} \to \mathcal{Y}$ so that*

$$Tf(\psi(\omega)) = \Phi(\omega, f(\omega)) \text{ for all } f \in A(\Omega, \mathcal{X}) \text{ and all } \omega \in \Omega.$$

In some instances, additional information concerning the functions ψ and Φ are known. For example, if $T : U(\Omega, \mathcal{X}) \to U(\Sigma, \mathcal{Y})$, then it can be shown that ψ is a uniform homeomorphism and Φ can be characterized. For details on this and for $\perp$-isomorphisms $T : \mathrm{Lip}(\Omega, \mathcal{X}) \to \mathrm{Lip}(\Sigma, \mathcal{Y})$, refer to [19].

Consider the space $C^p(\Omega, \mathcal{X})$, where $p \in \mathbb{N}$, Ω is an open set in a Banach space $\mathcal{Z}$ on which there is a C^p-bump function. It can be shown that if $f, g \in C^p(\Omega, \mathcal{X})$ satisfy $D^k f(\omega_0) = D^k g(\omega_0)$, $0 \le k \le p$, for some $\omega_0 \in \Omega$, then there exists $h \in C^p(\Omega, \mathcal{X})$ so that $\omega_0 \in \overline{\operatorname{int}[h = f]} \cap \overline{\operatorname{int}[h = g]}$. Therefore, we obtain the following counterpart of the preceding theorems for these spaces. For $k \in \mathbb{N}$, let $\mathcal{S}^k(\mathcal{Z}, \mathcal{X})$ be the space of all bounded symmetric k-linear operators from $\mathcal{Z}$ to $\mathcal{X}$.

Theorem 3.12. *Let $p, q \in \mathbb{N}$, Ω, Σ be open sets in a Banach spaces on which there are C^p, respectively, C^q-bump functions. Suppose that $T : C^p(\Omega, \mathcal{X}) \to C^q(\Sigma, \mathcal{Y})$ is a $\perp$-isomorphism. Denote by $\mathcal{Z}$ the Banach space containing Ω. Then there exist a homeomorphism $\psi : \Omega \to \Sigma$ and a function $\Phi : \Omega \times \mathcal{X} \times \mathcal{S}^1(\mathcal{Z}, \mathcal{X}) \times \cdots \times \mathcal{S}^p(\mathcal{Z}, \mathcal{X}) \to \mathcal{Y}$ so that*

$$Tf(\psi(\omega)) = \Phi(\omega, f(\omega), Df(\omega), \cdots, D^p f(\omega)), \ \ f \in C^p(\Omega, \mathcal{X}), \, \omega \in \Omega.$$

4. Applications

We present several applications of the results in §3.

4.1. Order isomorphism

In this subsection, let Ω, Σ be regular topological spaces and let X, Y be totally ordered sets endowed with the order topology, unless otherwise stated. Given subsets $A(\Omega, X), A(\Sigma, Y)$ of $C(\Omega, X)$ and $C(\Sigma, Y)$ respectively, an *order isomorphism* is a bijection $T : A(\Omega, X) \to A(\Sigma, Y)$ that preserves the pointwise order: for all $f, g \in A(\Omega, X)$,

$$f(\omega) \leq g(\omega) \text{ for all } \omega \in \Omega \iff Tf(\sigma) \leq Tg(\sigma) \text{ for all } \sigma \in \Sigma.$$

If $A(\Omega, X)$ and $A(\Sigma, Y)$ are lattices (in the pointwise order), then an order isomorphism is a lattice isomorphism. Following [29], we say that $A(\Omega, X)$ is X-*normal* if for any disjoint closed sets F_1, F_2 in Ω and any $x_1, x_2 \in X$, there exists $f \in A(\Omega, X)$ so that $f = x_i$ on F_i, $i = 1, 2$. The following statement is Kaplansky's Theorem in its full generality.

Theorem 4.1. *(Kaplansky [29]) Let Ω, Σ be compact Hausdorff spaces and let X, Y be totally ordered sets with the order topology. If $C(\Omega, X)$ and $C(\Sigma, Y)$ are X- and Y-normal respectively and there exists a lattice isomorphism $T : C(\Omega, X) \to C(\Sigma, Y)$, then Ω and Σ are homeomorphic.*

We will see that Kaplansky's Theorem as well as similar results on other function spaces can be derived from considerations in §3. A function $f \in C(\Omega, X)$ is *bounded* if there are $x_1, x_2 \in X$ so that $x_1 \leq f(\omega) \leq x_2$ for all $\omega \in \Omega$. Clearly, if Ω is compact Hausdorff, or if X has both largest and smallest elements, then all functions in $C(\Omega, X)$ are bounded.

Lemma 4.2. *Let Ω be a regular topological space and let X be a totally ordered set with the order topology. Suppose that $A(\Omega, X)$ is a X-normal sublattice of $C(\Omega, X)$ that consists of bounded functions. Then $A(X, E)$ satisfies condition (L).*

Proof. Let $h_1, h_2 \in A(\Omega, X)$, $U \in \mathrm{RO}(\Omega)$ and $\omega \notin \overline{U}$. Since Ω is regular, there exists $V \in \mathrm{RO}(\Omega)$ containing ω so that $\overline{V} \cap \overline{U} = \emptyset$. There are $x_1, x_2 \in X$ so that $x_1 \leq h_1(\omega), h_2(\omega) \leq x_2$ for all $\omega \in \Omega$. By X-normality, there are $k_1, k_2 \in A(\Omega, X)$ so that

$$k_1 = \begin{cases} x_2 & \text{on } \overline{V} \\ x_1 & \text{on } \overline{U} \end{cases} \quad \text{and} \quad k_2 = \begin{cases} x_1 & \text{on } \overline{V} \\ x_2 & \text{on } \overline{U} \end{cases}.$$

Set $k = (k_2 \vee h_1) \wedge (k_1 \vee h_2)$. Then $k \in A(\Omega, X)$. It is easy to see that $k = h_1$ on V and $k = h_2$ on U. This completes the verification of condition (L). $\square$

Proposition 4.3. *Let Ω, Σ be regular topological spaces and let X, Y be totally ordered sets with the order topology. Suppose that $A(\Omega, X)$ is a sublattice of $C(\Omega, X)$ that satisfies condition (L). Similarly for $A(\Sigma, Y)$. If $T : A(\Omega, X) \to A(\Sigma, Y)$ is a lattice isomorphism, then T is a $\perp$-isomorphism.*

Proof. Let $f, g, h \in A(\Omega, X)$ and suppose that $f \perp_h g$ and that $f, g \geq h$. Then $f \wedge g = h$ and hence $Tf \wedge Tg = Th$; whence $Tf \perp_{Th} Tg$. Similarly, $Tf \perp_{Th} Tg$ if $f \perp_h g$ and $f, g \leq h$.

Claim. If $f, g, h \in A(\Omega, X)$, $f \perp_h g$ and $f \geq h \geq g$, then $Tf \perp_{Th} Tg$.

Otherwise, there exists $\sigma \in \Sigma$ so that $Tf(\sigma) > Th(\sigma) > Tg(\sigma)$. Let $U = \sigma_{Tg}(Th)$. By condition (L), there exists $k \in A(\Sigma, Y)$ so that $k(\sigma) = Tf(\sigma)$ and $k = Th = Tg$ on $\operatorname{int} U^c$. Replace k by $(k \vee Th) \wedge Tf$ if necessary to assume additionally that $Th \leq k \leq Tf$.

If $\sigma_g(T^{-1}k) \not\subseteq \overline{\sigma_g(h)}$, there exist a nonempty $W \in \mathrm{RO}(\Omega)$ and $l \in A(\Omega, X)$ so that $W \subseteq \sigma_g(T^{-1}k)$, $l = g$ on $\overline{\sigma_g(h)}$ and $l = T^{-1}k$ on W. Replace l by $l \vee g$ if necessary so that $l \geq g$. (Note that $g \leq h \leq T^{-1}k \leq f$.) Then $l, h \geq g$ and $l \perp_g h$. Hence $Tl \perp_{Tg} Th$. Since $k = Tg$ on $\operatorname{int} U^c$, $\sigma_{Tg}(k) \subseteq U$. Thus

$$\sigma_{Tg}(Tl) \cap \sigma_{Tg}(k) \subseteq \sigma_{Tg}(Tl) \cap U = \sigma_{Tg}(Tl) \cap \sigma_{Tg}(Th) = \emptyset.$$

So $Tl \perp_{Tg} k$. Since $l, T^{-1}k \geq g$ as well, $l \perp_g T^{-1}k$. But $l = T^{-1}k$ on W. Hence $T^{-1}k = g$ on W, which is absurd since W is a nonempty subset of $\sigma_g(T^{-1}k)$. This shows that $\sigma_g(T^{-1}k) \subseteq \overline{\sigma_g(h)}$ and thus $\sigma_g(T^{-1}k) \subseteq \sigma_g(h)$.

By assumption, $\sigma_h(f) \cap \sigma_g(h) = \sigma_h(f) \cap \sigma_h(g) = \emptyset$. Therefore, $\sigma_g(T^{-1}k) \cap \sigma_h(f) = \emptyset$. If $\omega \in \sigma_g(T^{-1}k)$, then $\omega \notin \sigma_h(f)$ and hence $f(\omega) = h(\omega)$. So $T^{-1}k(\omega) = h(\omega)$ since $f \geq T^{-1}k \geq h$. On the other hand, if $\omega \notin \sigma_g(T^{-1}k)$, then $g(\omega) = T^{-1}k(\omega)$ and hence $T^{-1}k(\omega) = h(\omega)$ since $T^{-1}k \geq h \geq g$. Combining the two cases, we see that $T^{-1}k = h$ and hence $k = Th$. This is impossible since they differ at σ. This completes the proof of the claim.

Finally, let $f, g, h \in A(\Omega, X)$ with $f \perp_h g$. Then $f \diamond h \perp_h g \diamond h$, where each $\diamond$ stands for one of the symbols (not necessarily the same) $\vee$ or $\wedge$. By the first paragraph and the Claim, $T(f \diamond h) \perp_{Th} T(g \diamond h)$. Since

$$\begin{aligned} [Tf \neq Th] &= [(Tf \vee Th) \neq Th] \cup [(Tf \wedge Th) \neq Th] \\ &= [T(f \vee h) \neq Th] \cup [T(f \wedge h) \neq Th] \quad \text{and} \\ [Tg \neq Th] &= [T(g \vee h) \neq Th] \cup [T(g \wedge h) \neq Th], \end{aligned}$$

we see that $[Tf \neq Th] \cap [Tg \neq Th] = \emptyset$, i.e., $Tf \perp_{Th} Tg$. By symmetry, $Tf \perp_{Th} Tg$ implies $f \perp_h g$. This completes the proof of the proposition. □

The next result generalizes Kaplansky's Theorem and follows immediately from Theorem 3.7, Lemma 4.2 and Proposition 4.3.

Theorem 4.4. *(See also* [16]*.) Let Ω, Σ be compact Hausdorff spaces and let X, Y be totally ordered sets with the order topology. If $A(\Omega, X)$ is a X-normal sublattice of $C(\Omega, X)$, $A(\Sigma, Y)$ is a Y-normal sublattice of $C(\Sigma, Y)$ and there is a lattice isomorphism $T : A(\Omega, X) \to A(\Sigma, Y)$, then Ω and Σ are homeomorphic.*

Proposition 4.3 and Theorem 3.11 also yield the following.

Theorem 4.5. *Let $A(\Omega)$ be one of the spaces of real valued functions $U(\Omega)$, $U_*(\Omega)$, $\mathrm{Lip}(\Omega)$ or $\mathrm{Lip}_*(\Omega)$, where Ω is a complete metric space. Similarly for $A(\Sigma)$. If $T : A(\Omega) \to A(\Sigma)$ is a lattice isomorphism, then there are a homeomorphism $\psi : \Omega \to \Sigma$ and a function $\Phi : \Omega \times \mathbb{R} \to \mathbb{R}$ so that*

$$Tf(\psi(\omega)) = \Phi(\omega, f(\omega)) \text{ for all } f \in A(\Omega) \text{ and all } \omega \in \Omega.$$

Linear and nonlinear lattice and order isomorphisms have been well studied in a variety of function spaces. Garrido and Jaramillo [21] showed that the unital vector lattices $U(\Omega)$ and $U_*(\Omega)$ determine Ω up to uniform homeomorphism. In [22], the same authors showed that as a unital vector lattice, $\mathrm{Lip}(\Omega)$ determines Ω up to Lipschitz homeomorphism. For Lipschitz spaces defined on Banach spaces, F. and J. Cabello Sánchez showed that $\mathrm{Lip}(\mathbb{R})$ and $\mathrm{Lip}_*(\mathbb{R})$ are isomorphic as vector lattices. However, if $\mathcal{X}$ is a Banach space of dimension > 1 and $\mathcal{Y}$ is a Banach space, then $\mathrm{Lip}_*(\mathcal{X})$ is not isomorphic as a vector lattice to $\mathrm{Lip}(\mathcal{Y})$ [15].

As a lattice alone (i.e., disregarding linearity), Shirota [39] proved that if $U_*(\Omega)$ and $U_*(\Sigma)$ are lattice isomorphic, with Ω, Σ complete metric spaces, then Ω is uniformly homeomorphic to Σ. In the same paper, the claim was also made for lattice isomorphisms $T : U(\Omega) \to U(\Sigma)$; but the proof contains a gap. The gap was repaired by F. Cabello Sánchez [12] and F. and J. Cabello Sánchez [14]. The same authors also showed that if $T : C^p(\Omega) \to C^p(\Sigma)$ is an order isomorphism, where $p \in \mathbb{N} \cup \{\infty\}$ and Ω, Σ are manifolds modeled on Banach spaces that support C^p-bump functions, then Ω and Σ are homeomorphic [13]. A unified treatment of order isomorphisms between functions spaces can be found in [34].

4.2. Realcompact spaces

One can also consider the situation for Theorem 4.4 away from the confines of compact Hausdorff spaces. A completely regular Hausdorff space Ω has a "largest" compactification, the *Stone-Čech compactification* $\beta\Omega$, characterized by the fact that every continuous function f from Ω into a compact Hausdorff space X has a unique continuous extension $\widehat{f} : \beta\Omega \to X$. A good source of information concerning the Stone-Čech compactification is [42]. For the purpose of extending the Gelfand-Kolmogorov Theorem, Hewitt [27] introduced the class of realcompact spaces. Let $\mathbb{R}_\infty$ be the one point compactification of $\mathbb{R}$. The (Hewitt) *realcompactification* $\upsilon\Omega$ consists of all $\omega_0 \in \beta\Omega$ such that for any continuous real-valued function f on Ω, its continuous extension $\widehat{f} : \beta\Omega \to \mathbb{R}_\infty$ satisfies $\widehat{f}(\omega_0) \in \mathbb{R}$. Ω is *realcompact* if $\Omega = \upsilon\Omega$. It is known that a space is realcompact if and only if it is homeomorphic to a closed subspace of $\mathbb{R}^\Gamma$ for some index set Γ; see, e.g., [23]. Hewitt showed that for realcompact spaces, $C(\Omega)$ as a ring determines Ω uniquely up to homeomorphism. The result was generalized by Araujo, Beckenstein and Narici [5], and subsequently by Araujo to vector valued functions [1, 2]. If $\mathcal{X}$ and $\mathcal{Y}$ are vector spaces, denote the set of all linear bijections from $\mathcal{X}$ onto $\mathcal{Y}$ by $I(\mathcal{X}, \mathcal{Y})$.

Theorem 4.6. [2] *Let Ω and Σ be realcompact spaces and let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces. If $T : C(\Omega, \mathcal{X}) \to C(\Sigma, \mathcal{Y})$ is a linear biseparating map (i.e., linear $\perp$-isomorphism), then there are a homeomorphism $\varphi : \Sigma \to \Omega$ and a function $J : \Sigma \to I(\mathcal{X}, \mathcal{Y})$ so that*

$$Tf(\sigma) = (J\sigma)f(\varphi(\sigma)) \text{ for all } f \in C(\Omega, \mathcal{X}) \text{ and all } \sigma \in \Sigma.$$

Without the assumption of linearity in Theorem 4.6, it is still possible to conclude that Ω and Σ are homeomorphic. But the representation of T may not hold.

Theorem 4.7. [19] *Let Ω, Σ be realcompact spaces and $\mathcal{X}, \mathcal{Y}$ be Hausdorff topological vector spaces. If $T : C(\Omega, \mathcal{X}) \to C(\Sigma, \mathcal{Y})$ is a $\perp$-isomorphism, then Ω and Σ are homeomorphic.*

In particular, using the arguments of Lemma 4.2 and Proposition 4.3, the result can be applied to lattice isomophisms. We emphasize that the lattice isomorphism below need not be linear.

Theorem 4.8. [34] *Let Ω and Σ be realcompact spaces. If there is a lattice isomorphism $T : C(\Omega) \to C(\Sigma)$, then Ω and Σ are homeomorphic.*

4.3. Ring isomorphism and multiplicative isomorphism

Let Ω be a Hausdorff topological space. The space $C(\Omega)$ of all real valued continuous functions on Ω is a (unital) ring under pointwise operations. The subring $C_*(\Omega)$ consists of the bounded functions. Clearly, the (pointwise) order on these rings is determined by the ring structure since $f \geq 0$ if and only if f is a square in the ring. It follows immediately that results from §4.1 give rise to corresponding results concerning ring isomorphisms. In particular, we cite Hewitt's generalization of the theorem of Gelfand-Kolmogorov as a consequence of Theorem 4.8.

Theorem 4.9. [27] *Let Ω and Σ be realcompact spaces. Suppose that $C(\Omega)$ and $C(\Sigma)$ are isomorphic as rings, then Ω and Σ are homeomorphic.*

If Ω is a complete metric space, then $\mathrm{Lip}_*(\Omega)$ is a ring under pointwise operations. Ring isomorphisms between such rings were described in [22]. Let Ω, Σ be metric spaces. A function $\psi : \Omega \to \Sigma$ is *Lipschitz in the small* if there exist $r, K > 0$ so that $d(\psi(\omega_1), \psi(\omega_2)) \leq K d(\omega_1, \omega_2)$ whenever $\omega_1, \omega_2 \in \Omega$ and $d(\omega_1, \omega_2) < r$. ψ is a *LS-homeomorphism* if it is a homeomorphism so that both ψ and ψ^{-1} are Lipschitz in the small.

Theorem 4.10. *(Garrido and Jaramillo) Let Ω, Σ be complete metric spaces. The following are equivalent.*

1. $\mathrm{Lip}_*(\Omega)$ *and* $\mathrm{Lip}_*(\Sigma)$ *are isomorphic as unital rings.*
2. $\mathrm{Lip}_*(\Omega)$ *and* $\mathrm{Lip}_*(\Sigma)$ *are isomorphic as unital vector lattices.*
3. Ω *and* Σ *are LS-homeomorphic.*

Garrido, Jaramillo and Prieto showed that the ring of smooth functions $C^\infty(M)$ determines the manifold M up to smooth diffeomorphism. For notions and notation regarding global analysis on infinite dimensional manifolds, refer to [30].

Theorem 4.11. *(Garrido, Jaramillo and Prieto* [24]*) Let M and N be paracompact Banach manifolds modeled on C^∞-smooth Banach spaces. The rings $C^\infty(M)$ and $C^\infty(N)$ are isomorphic if and only if M and N are C^∞-diffeomorphic.*

Instead of ring isomorphisms, one can disregard linearity and consider maps that preserve multiplication alone.

Proposition 4.12. *Let Ω, Σ be Hausdorff spaces and let $A(\Omega), A(\Sigma)$ be unital subrings of $C(\Omega)$ and $C(\Sigma)$ respectively. Assume that either*

1. *Ω and Σ are compact and $A(\Omega), A(\Sigma)$ satisfy condition (L); or*
2. *Ω and Σ are complete metric spaces and $A(\Omega), A(\Sigma)$ satisfy conditions (L) and (L_s).*

If $T : A(\Omega) \to A(\Sigma)$ is a multiplicative isomorphism, i.e., T is a bijection so that $T(fg) = Tf \cdot Tg$ for all $f, g \in A(\Omega)$, then there is a homeomorphism $\psi : \Omega \to \Sigma$ that is associated with T in the sense defined before Theorem 3.7. *In particular, T is a $\perp$-isomorphism.*

Proof. By assumption, the constant functions belong to $A(\Omega)$ and $A(\Sigma)$. Let Let $0, 2$ denote the constant functions with values $0, 2$ respectively. Then

$$2 \cdot T0 = T(T^{-1}2 \cdot 0) = T0.$$

Hence $T0 = 0$. For any $f, g \in C(\Omega)$,

$$f \perp_0 g \iff fg = 0 \iff Tf \cdot Tg = T0 = 0 \iff Tf \perp_{T0} Tg.$$

Hence T is a $\perp_0$-isomorphism. Note that Ω is a regular topological space and that $A(\Omega)$ is nowhere trivial and satisfies condition (L). Hence $A(\Omega)$ is weakly regular. Similarly for $A(\Sigma)$. By Theorem 3.2, there is a Boolean isomorphism $\theta_0 : \mathrm{RO}(\Omega) \to \mathrm{RO}(\Sigma)$ associated with $(T, 0)$. The same argument from the proof of Theorem 3.7 or Theorem 3.8 shows that Proposition 3.6 applies to θ_0. Thus there is a homeomorphism $\psi : \Omega \to \Sigma$ so that for any $f \in C(\Omega)$ and any $U \in \mathrm{RO}(\Omega)$, $f = 0$ on U if and only if $Tf = 0$ on $\psi(U)$.

In fact, ψ is associated with T. Let $U \in \mathrm{RO}(\Omega)$ and let $f, g \in A(\Omega)$ be such that $f = g$ on U. For $\sigma \in \psi(U)$, it follows from condition (L) that there exists $h \in C(\Sigma)$ so that $h(\sigma) = 1$ and $h = 0$ on $\operatorname{int} \psi(U)^c = \psi(\operatorname{int} U^c)$. Hence $T^{-1}h = T^{-1}0 = 0$ on $\operatorname{int} U^c$. Thus $T^{-1}h \cdot f = T^{-1}h \cdot g$. Therefore, $h \cdot Tf = h \cdot Tg$. In particular, $Tf(\sigma) = Tg(\sigma)$. This proves that $Tf = Tg$ on $\psi(U)$ if $f = g$ on U. The reverse implication follows by symmetry.

Let $\theta : \mathrm{RO}(\Omega) \to \mathrm{RO}(\Sigma)$ be the Boolean isomorphism given by $\theta(U) = \psi(U)$. Then θ is associated with T. Hence T is a $\perp$-isomorphism by Theorem 3.3. □

Proposition 4.12 and Theorem 3.11 give.

Theorem 4.13. *Let Ω, Σ be complete metric spaces and let $A(\Omega)$ be one of the spaces $U_*(\Omega)$ or $\mathrm{Lip}_*(\Omega)$. Similarly for $A(\Sigma)$. Let $T : A(\Omega) \to A(\Sigma)$ be a multiplicative isomorphism. Then there are a homeomorphism $\psi : \Omega \to \Sigma$ and a function $\Phi : \Omega \times \mathbb{R} \to \mathbb{R}$ so that*

$$Tf(\psi(\omega)) = \Phi(\omega, f(\omega)) \textit{ for all } f \in A(\Omega) \textit{ and all } \omega \in \Omega.$$

Milgram [37] characterized all multiplicative isomorphisms $T : C(\Omega) \to C(\Sigma)$. A combination of Proposition 4.12 and Theorems 3.7, 3.10 gives a partial result in this regard. See [16] for a proof of Milgram's Theorem via

$\perp$-isomorphisms. When $p \in \mathbb{N}$ and Ω is a C^p-manifold, Mrčun and Šemrl [38] showed that all multiplicative automorphisms T on $C^p(\Omega)$ are of the form $Tf = f \circ \psi$ for some C^p diffeomorphisms ψ. The result was extended to the case $p = \infty$ by Artstein-Avidan, Faifman and Milman [9]. See [31] for a survey on the multiplication operator and other operator functional equations.

4.4. Isometry

The study of isometries is probably the most well developed part among theorems of Banach-Stone type. Here we restrict ourselves to a much abridged survey. Further information can be found in the two-volume monograph [20].

Behrends [11] introduced the use of centralizers into Banach-Stone considerations. Let $\mathcal{X}$ be a Banach space and denote the set of extreme points of the ball in $\mathcal{X}^*$ by $\operatorname{ext}\mathcal{X}^*$. A bounded linear operator $S : \mathcal{X} \to \mathcal{X}$ is a *multiplier* if every $x^* \in \operatorname{ext}\mathcal{X}^*$ is an eigenvector of S^*, i.e., $S^*x^* = a_S(x^*)x^*$ for some scalar $a_S(x^*)$. If R, S are multipliers, say that R is an *adjoint* of S if $a_R(x*) = \overline{a_S(x^*)}$ for all $x^* \in \mathcal{X}^*$. The *centralizer* $Z(\mathcal{X})$ of $\mathcal{X}$ consists of all multipliers S for which an adjoint exists. Note that for real Banach spaces, the centralizer is the same as the set of all multipliers. Multiples of the identity operator are always present in the centralizer. Say that $\mathcal{X}$ has *trivial centralizer* if there are no other operators in $Z(\mathcal{X})$. Many classes of Banach spaces have trivial centralizers; refer to [11, 20].

Theorem 4.14. *(Behrends) Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces which have trivial centralizers. Suppose further that Ω and Σ are locally compact Hausdorff spaces and that there exists a surjective linear isometry $T : C_0(\Omega, \mathcal{X}) \to C_0(\Sigma, \mathcal{Y})$. Then there is a homeomorphism $\varphi : \Sigma \to \Omega$ and a continuous function V from Σ into the space of isometries from $\mathcal{X}$ onto $\mathcal{Y}$ (given the strong operator topology) such that*

$$Tf(\sigma) = V(\sigma)f(\varphi(\sigma)) \text{ for all } f \in C_0(\Omega, \mathcal{X}) \text{ and all } \sigma \in \Sigma.$$

Araujo [4] extended this result by way of finding a connection to $\perp$-isomorphisms (biseparating maps).

Theorem 4.15. *(Araujo) Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces which have trivial centralizers. Assume one of the following situations.*

1. *Ω, Σ are realcompact spaces and $\mathcal{X}, \mathcal{Y}$ are infinite dimensional. $A(\Omega, \mathcal{X}) = C_*(\Omega, \mathcal{X})$, the space of bounded $\mathcal{X}$-valued continuous functions on Ω, with the sup-norm. $A(\Sigma, \mathcal{Y}) = C_*(\Sigma, \mathcal{Y})$.*
2. *Ω, Σ are complete metric spaces, $A(\Omega, \mathcal{X}) = U_*(\Omega, \mathcal{X})$, the space of bounded $\mathcal{X}$-valued uniformly continuous functions on Ω. $A(\Sigma, \mathcal{Y}) = U_*(\Sigma, \mathcal{Y})$.*

If $T : A(\Omega, \mathcal{X}) \to A(\Sigma, \mathcal{Y})$ is a surjective linear isometry, then it is a $\perp$-isomorphism. Consequently, there is a homeomorphism $\varphi : \Sigma \to \Omega$ and a continuous function V from Σ into the space of isometries from $\mathcal{X}$ onto $\mathcal{Y}$ (given the strong operator topology) such that

$$Tf(\sigma) = V(\sigma)f(\varphi(\sigma)) \text{ for all } f \in C_0(\Omega, \mathcal{X}) \text{ and all } \sigma \in \Sigma.$$

In case (2), φ is a uniform homeomorphism.

Let Ω be a complete metric space and let $\mathcal{X}$ be a Banach space. The space of $\mathcal{X}$-valued Lipschitz functions on Ω, $\mathrm{Lip}(\Omega, \mathcal{X})$ is a Banach space under the norm

$$\|f\| = \max\{\|f\|_\infty, L(f)\},$$

where

$$\|f\|_\infty = \sup_{\omega\in\Omega} \|f(\omega)\| \text{ and } L(f) = \sup_{\omega_1\neq\omega_2} \frac{\|f(\omega_1) - f(\omega_2)\|}{d(\omega_1, \omega_2)}.$$

Araujo and Dubarbie [6] gave a complete description of isometries between vector-valued spaces of Lipschitz functions. We state a special case of their result here. Define an equivalence relation on Ω by $x \sim y$ if there are $x = x_1, \ldots, x_n = y$ in Ω so that $d(x_i, x_{i+1}) < 2$, $1 \le i < n$. The equivalence classes are called the 2-*components* of Ω.

Theorem 4.16. *Let Ω, Σ be complete metric spaces and let $\mathcal{X}, \mathcal{Y}$ be Banach spaces. Assume that $T : \mathrm{Lip}(\Omega, \mathcal{X}) \to \mathrm{Lip}(\Sigma, \mathcal{Y})$ is a surjective linear isometry so that for all $\sigma \in \Sigma$, there is a constant function $f \in \mathrm{Lip}(\Omega, \mathcal{X})$ so that $Tf(\sigma) \neq 0$. Then there is a homeomorphism $\varphi : \Sigma \to \Omega$ and a continuous function V from Σ into the space of isometries from $\mathcal{X}$ onto $\mathcal{Y}$ (given the strong operator topology) such that*

$$Tf(\sigma) = V(\sigma)f(\varphi(\sigma)) \text{ for all } f \in \mathrm{Lip}(\Omega, \mathcal{X}) \text{ and all } \sigma \in \Sigma.$$

Moreover, V is constant on each 2-component of Σ and $d_{\mathcal{Y}}(\varphi(\omega_1), \varphi(\omega_2)) = d_{\mathcal{X}}(\omega_1, \omega_2)$ if either of these quantities is < 2.

It is worth mentioning that in the course of the proof of Theorem 4.16, it is first shown that T is a $\perp$-isomorphism (biseparating). Characterization of linear isometries on certain spaces of scalar-valued Lipschitz functions was obtained earlier by Weaver [43].

4.5. Nonvanishing preservers

In this part, assume that Ω, Σ are regular topological spaces and X, Y are Hausdorff spaces. Let $A(\Omega, X)$ and $A(\Sigma, Y)$ be subsets of $C(\Omega, X)$ and $C(\Sigma, Y)$ respectively. Given $n \in \mathbb{N}$ and $h \in A(\Omega, X)$, a bijection $T : A(\Omega, X) \to A(\Sigma, Y)$ is a $\cap_h^n$-*isomorphism* if for any $f_1, \ldots, f_n \in A(\Omega, X)$,

$$\bigcap_{i=1}^{n} [f_i = h] = \emptyset \quad \Longleftrightarrow \quad \bigcap_{i=1}^{n} [Tf_i = Th] = \emptyset.$$

T is a $\cap^n$-*isomorphism* if it is a $\cap_h^n$-isomorphism for all $h \in A(\Omega, X)$. It is clear that every $\cap_h^n$-isomorphism is a $\cap_h^m$-isomorphism if $n > m$. Hence every $\cap^n$-isomorphism is a $\cap^m$-isomorphism if $n > m$. $\cap^n$-isomorphisms were introduced by Hernández and Ródenas [26]. Further results were given in [17, 33].

Proposition 4.17. *Let Ω, Σ be regular topological spaces. Suppose that $A(\Omega, X)$ and $A(\Sigma, Y)$ satisfy condition (L) and that there exists $k \in A(\Omega, X)$ so that $[k = h] = \emptyset$. If $T : A(\Omega, X) \to A(\Sigma, Y)$ is a $\cap_h^2$-isomorphism, then it is a $\perp_h$-isomorphism.*

Proof. First of all, since T is a $\cap_h^2$-isomorphism, it is a $\cap_h^1$-isomorphism. Thus $[k = h] = \emptyset$ implies $[Tk = Th] = \emptyset$. Suppose that there are $f, g \in A(X, E)$ so that $f \perp_h g$ but $Tf \not\perp_{Th} Tg$. There exists $\sigma_0 \in \Sigma$ where $Tf(\sigma_0), Tg(\sigma_0) \neq Th(\sigma_0)$. Since Σ is a regular toopological space, there exists $V \in \mathrm{RO}(\Sigma)$ containing σ_0 so that $V \subseteq [Tf \neq Th] \cap [Tg \neq Th]$. Then $\sigma_0 \notin \overline{\operatorname{int} V^c}$ and $\operatorname{int} V^c \in \mathrm{RO}(\Sigma)$. As $A(\Sigma, Y)$ satisfies condition (L), there exist $l \in A(\Sigma, Y)$ and $W \in \mathrm{RO}(Y)$ so that $\sigma_0 \in W$, $l = Tk$ on $\operatorname{int} V^c$ and $l = Th$ on W. Now

$$[Tf \neq Th] \cup [l \neq Th] \supseteq V \cup \overline{\operatorname{int} V^c} = \Sigma.$$

Thus $[Tf = Th] \cap [l = Th] = \emptyset$ and hence $[f = h] \cap [T^{-1}l = h] = \emptyset$. Similarly, $[g = h] \cap [T^{-1}l = h] = \emptyset$. But since $f \perp_h g$, $[f = h] \cup [g = h] = \Omega$. Therefore, $[T^{-1}l = h] = \emptyset$, whence $[l = Th] = \emptyset$, contradicting the fact that $l = Th$ on $W \neq \emptyset$. This completes the proof of the proposition. $\square$

The next two results follow easily from Proposition 4.17, Theorem 3.7 and Theorem 3.11.

Theorem 4.18. *Let Ω, Σ be compact Hausdorff spaces and let $\mathcal{X}, \mathcal{Y}$ be normed spaces. If $T : C(\Omega, \mathcal{X}) \to C(\Sigma, \mathcal{Y})$ is a $\cap^2$-isomorphism, then there is a homeomorphism $\psi : \Omega \to \Sigma$ associated with T.*

Theorem 4.19. *Let Ω, Σ be complete metric spaces and let $\mathcal{X}, \mathcal{Y}$ be normed spaces. Suppose that $A(\Omega, \mathcal{X})$ is one of the spaces $U(\Omega, \mathcal{X}), U_*(\Omega, \mathcal{X})$, $\mathrm{Lip}(\Omega, \mathcal{X})$, $\mathrm{Lip}_*(\Omega, \mathcal{X})$. Similarly for $A(\Sigma, \mathcal{Y})$. If $T : A(\Omega, \mathcal{X}) \to A(\Sigma, \mathcal{Y})$ is a $\cap^2$-isomorphism, then there are a homeomorphism $\psi : \Omega \to \Sigma$ and a function $\Phi : \Omega \times \mathcal{X} \to \mathcal{Y}$ so that*

$$Tf(\psi(\omega)) = \Phi(\omega, f(\omega)) \text{ for all } f \in A(\Omega, \mathcal{X}) \text{ and all } \omega \in \Omega.$$

In general, a $\cap^1$-isomorphism need not be a $\cap^2$-isomorphism, as the following example shows.

Example. Let $I = [0, 1]$. Define $T : C(I, I) \to C(I, I)$ by

$$Tf = \begin{cases} 1 - f & \text{if range } f = [0, 1] \\ f & \text{otherwise.} \end{cases}$$

Then T is a $\cap^1$-isomorphism but not a $\cap^2$-isomorphism, nor is T is a $\perp$-isomorphism.

Indeed, it is easy to check that if $f, g \in C(I, I)$, then $[f = g] \neq \emptyset$ if and only if $[Tf = Tg] = \emptyset$. However, let f be the constant function with value $\frac{1}{4}$ and let $g \in C(I, I)$ be such that $g(\frac{1}{4}) \neq \frac{1}{4} = g(\frac{3}{4})$ and $\operatorname{range} g \neq [0, 1]$. Let h be the identity function $h(t) = t$ for all $t \in I$. Then $[f = h] \cap [g = h] = \emptyset$ but

$$\frac{3}{4} \in [f = 1 - h] \cap [g = 1 - h] = [Tf = Th] \cap [Tg = Th].$$

Hence T is not a $\cap^2$-isomorphism.

To see that T is not a $\perp$-isomorphism, consider the same h but take $f = h \vee \frac{1}{2}$, $g = h \wedge \frac{1}{2}$. It is clear that $f \perp_h g$ but that $Tf \not\perp_{Th} Tg$.

However, Li and Wong [35, 36] obtained a number of results regarding *linear* $\cap^1$-isomorphisms. The theorem below gives some special cases of their results.

Theorem 4.20. *(Li and Wong) Let Ω, Σ be Hausdorff completely regular topological spaces and let $\mathcal{X}, \mathcal{Y}$ be normed spaces. Assume that $A(\Omega, \mathcal{X})$ is the space $C(\Omega, \mathcal{X})$, or, where Ω is complete metric, $U(\Omega, \mathcal{X})$ or* $\mathrm{Lip}(\Omega, \mathcal{X})$. *Similarly for $A(\Sigma, \mathcal{Y})$. If $T : A(\Omega, \mathcal{X}) \to A(\Sigma, \mathcal{Y})$ is a linear $\cap^1$-isomorphism, then it is a $\perp$-isomorphism.*

We close with a positive result concerning nonlinear $\cap^1$-isomorphisms. Let Ω, Σ be Hausdorff spaces. We call a bijection $T : C(\Omega) \to C(\Sigma)$ an *anti-order isomorphism* if $f \geq g \iff Tg \geq Tf$ for all $f, g \in C(\Omega)$. Evidently, T is an anti-isomorphism if and only if the operator $-T$ is an order isomorphism, where $(-T)f := -Tf$.

Theorem 4.21. *Let Ω, Σ be connected compact Hausdorff spaces. If $T : C(\Omega) \to C(\Sigma)$ is a $\cap^1$-isomorphism, then T is an order isomorphism or an anti-order isomorphism. In particular, T is a $\perp$-isomorphism and hence Ω and Σ are homeomorphic.*

Proof. Since Ω is connected, given any two functions $h, k \in C(\Omega)$ with $[h = k] = \emptyset$, either $h > k$ (i.e., $h(\omega) > k(\omega)$ for all ω) or $k > h$. Similarly for $C(\Sigma)$. We break the proof of the theorem into a series of steps.

Claim 1. If $f, g \in C(\Omega)$ and $f \leq g$, then either $Tf \leq Tg$ or $Tg \leq Tf$.

Otherwise, there are $\sigma_1, \sigma_2 \in \Sigma$ such that $Tf(\sigma_1) > Tg(\sigma_1)$ and $Tg(\sigma_2) > Tf(\sigma_2)$. Let $k_i \in C(\Sigma)$ be functions such that $k_2 > Tf > k_1$ and $k_i(\sigma_i) = Tg(\sigma_i)$, $i = 1, 2$. For $i = 1, 2$, $[T^{-1}k_i = f] = \emptyset$ and $[T^{-1}k_i = g] \neq \emptyset$. By the statement before Claim 1, $T^{-1}k_i > f$. Thus there exists $h \in C(\Omega)$ so that $h > f$ and $[h = T^{-1}k_i] \neq \emptyset$, $i = 1, 2$. But then $[Th = Tf] = \emptyset$ and $[Th = k_i] \neq \emptyset$, $i = 1, 2$; hence $Tf \not< Th$ and $Th \not< Tf$, contrary to the statement before Claim 1.

Claim 2. If $f, g, h \in C(\Omega)$ and $f \leq g, h$, then either $Tf \leq Tg, Th$ or $Tg, Th \leq Tf$.

If either of g, h equals f, then Claim 2 follows from Claim 1. Otherwise, we may choose $\omega_1, \omega_2 \in \Omega$ so that $g(\omega_1) > f(\omega_1)$ and $h(\omega_2) > f(\omega_2)$. Let $k \in C(\Omega)$ be such that $k > f$ and $k(\omega_1) = g(\omega_1)$, $k(\omega_2) = h(\omega_2)$. By the first statement of the proof, either $Tk > Tf$ or $Tk < Tf$. Assume the former. By Claim 1, either $Tg \geq Tf$ or $Tg \leq Tf$. Since $[Tg = Tk] \neq \emptyset$, we must have $Tg \geq Tf$. Similarly $Th \geq Tf$. If $Tk < Tf$, then we can show analogously that $Tg, Th \leq Tf$.

The following variant of Claim 2 can be established in the same way: if $g, h \leq f$, then either $Tg, Th \leq Tf$ or $Tf \leq Tg, Th$.

Claim 3. Let $f \in C(\Omega)$. Then either

$$g \le f \le h \implies Tg \le Tf \le Th \text{ or}$$
$$g \le f \le h \implies Th \le Tf \le Tg$$

In the first case, we say that T is order preserving with respect to f and in the second case, T is anti-order preserving with respect to f.

Otherwise, there are $g, h \neq f$, $g \le f \le h$ so that either $Tg, Th \ge Tf$ or $Tg, Th \le Tf$. Apply Claim 2 or its variant to T^{-1} to see that either $g, h \ge f$ or $g, h \le f$, contrary to the choices of g and h.

We now show that either T is an order isomorphism or an anti-order isomorphism. Otherwise, taking symmetry into account, by Claim 3, we may assume that there are h_1, h_2 so that T is order preserving with respect to h_1 and anti-order preserving with respect to h_2. Since T is a bijection, $h_1 \neq h_2$. Now $h_1 \wedge h_2 \le h_1, h_2$. Thus $Th_2 \le T(h_1 \wedge h_2) \le Th_1$. On the other hand, by Claim 2, either $T(h_1 \wedge h_2) \le Th_1, Th_2$ or $T(h_1 \wedge h_2) \ge Th_1, Th_2$. Assume the former case; the proof is similar in the latter case. We have $Th_2 \le T(h_1 \wedge h_2) \le Th_2$. Hence $h_1 \wedge h_2 = h_2$, i.e., $h_2 \le h_1$. But since T is order preserving with respect to h_1 and anti-order preserving with respect to h_2, $Th_2 \le Th_1$ and $Th_1 \le h_2$. Thus $h_1 = h_2$, contrary to their choices. This concludes the proof that T is either an order isomorphism or anti-order isomorphism. Applying Proposition 4.3 to either T or $-T$, we see that T is a $\perp$-isomorphism. By Theorem 4.4, Ω and Σ are homeomorphic. $\square$

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Denny H. Leung
Department of Mathematics
National University of Singapore
Singapore 119076
e-mail: dennyhl@u.nus.edu

Wee Kee Tang
School of Physical and Mathematical Sciences
Nanyang Technological University
Singapore 637371
e-mail: weekeetang@ntu.edu.sg