FUNCTIONS THAT ARE LIPSCHITZ IN THE SMALL

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Abstract. Let X and Y be metric spaces. A function $f : X \to Y$ is said to be *Lipschitz in the small* if there are $r > 0$ and $K < \infty$ so that $d(f(u), f(v)) \leq Kd(u, v)$ for any $u, v \in X$ with $d(u, v) \leq r$. We find necessary and sufficient conditions on a subset A of X such that $f_{|A}$ is Lipschitz for every function f that is Lipschitz in the small on X. We also find necessary and sufficient conditions on X for $\mathrm{LS}(X)$ to be linearly order isomorphic to $\mathrm{Lip}(Y)$ for some metric space Y.

Let X and Y be metric spaces. We use the same symbol d to denote the metrics on both spaces. Let $f : X \to Y$ be a function. f is *Lipschitz* if there exists $K < \infty$ so that $d(f(u), f(v)) \leq Kd(u, v)$; f is said to be *Lipschitz in the small* (abbreviated LS) if there exist $r > 0$ and $K < \infty$ so that $d(f(u), f(v)) \leq Kd(u, v)$ if $d(u, v) \leq r$. The space of all functions from X to Y that are Lipschitz in the small is denoted by $\mathrm{LS}(X, Y)$. We abbreviate $\mathrm{LS}(X, \mathbb{R})$ to $\mathrm{LS}(X)$. Clearly every Lipschitz function is LS and every LS function is locally Lipschitz and uniformly continuous. The notion of Lipschitz in the small functions is due to J. Luukkainen [8] and later studied by various authors (see e.g., [1],[2],[4]). It has been shown in [8] that two complete metric spaces X and Y are LS homeomorphic (i.e., there is a bijection $h : X \to Y$ such that h and h^{-1} are LS) if and only if the algebras of bounded Lipschitz functions $\mathrm{Lip}_b(X)$ and $\mathrm{Lip}_b(Y)$ are isomorphic. This Banach-Stone type result has been extended in [4] to LS spaces. More precisely, X and Y are LS homeomorphic if and only if the corresponding function classes $\mathrm{LS}(X)$ and $\mathrm{LS}(Y)$ are isomorphic. An important fact about $\mathrm{LS}(X)$ is its uniform density in the uniformly continuous real-valued functions defined on an arbitrary metric space X, as shown by Garrido and Jaramillo [4], and then by Beer and Garrido [2] using very different methods. This parallels the uniform density of the real-valued locally Lipschitz functions in the real-valued continuous functions, usually shown using locally Lipschitz partitions of unity, as discussed for example in a subsequent paper of Beer and Garrido [3].

Let A be a subset of X. We say that A is a *Lipschitz set* for a class $\mathcal{F}(X, Y)$ of function from X to Y if $f_{|A}$ is a Lipschitz function with respect to the metric d for any function $f \in \mathcal{F}(X, Y)$. In [2, Theorems 4.2, 4.3], the

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authors found conditions for a set to be a Lipschitz set for locally Lipschitz functions and uniformly locally Lipschitz functions.

For any $\varepsilon > 0$ and any $x, y \in X$, an *ε-chain* from x to y is a finite sequence $x = x_0, x_1, \ldots, x_n = y$ in X so that $d(x_{k-1}, x_k) \leq \varepsilon$ for $1 \leq k \leq n$. We say that $x \sim_\varepsilon y$ if there is an ε-chain from x to y. The relation $\sim_\varepsilon$ is an equivalence relation, the equivalence classes of which are called *ε-step territories* of X. The following sufficient conditions for a set A to be a Lipschitz set for $\mathrm{LS}(X)$ are also known; see e.g., [2, p. 812 and Theorem 4.5]: (1) the restriction of d to $A \times A$ is an almost convex metric, and (2) A is a Bourbaki bounded subset. Recall that a subset A of X is *Bourbaki bounded* if for each $\varepsilon > 0$ there is a finite subset F of X and $n \in \mathbb{N}$ such that each point of A can be joined to some point of F by an ε-chain of length n and that $d_{|A\times A}$ is an *almost convex metric* if whenever $a_1 \neq a_2$ in A and α satisfies $d(a_1, a_2) < \alpha$, then for each $\beta \in (0, \alpha)$, there exists $a_3 \in A$ with $d(a_1, a_3) < \beta$ and $d(a_2, a_3) < \alpha - \beta$. The result [2, Theorem 4.5] cited above actually characterizes a set A so that $f_{|A}$ is both bounded and Lipschitz for every $f \in \mathrm{LS}(X, Y)$. It is also worthwhile to note that the Lipschitz sets while hereditary need not be stable under finite unions. The first main result of the present note is a characterization of Lipschitz sets for $\mathrm{LS}(X, Y)$. If E is an ε-step territory of X, define a metric d_ε on E by

$$d_\varepsilon(x, y) = \inf\{\sum_{k=1}^{n} d(x_{k-1}, x_k) : (x_k)_{k=0}^{n} \text{ is an } \varepsilon\text{-chain from } x \text{ to } y\}$$

for all $x, y \in E$. As far as we know, the terminology of ε-step territories and use of the metrics d_ε were first made by O'Farrell [9]. It is clear that $d(x, y) \leq d_\varepsilon(x, y)$ for all $x, y \in E$ and that $d(x, y) = d_\varepsilon(x, y)$ if $d(x, y) \leq \varepsilon$.

Theorem 1. *Let A be a subset of a metric space X. The following are equivalent.*

(1) *The set A is a Lipschitz set for $\mathrm{LS}(X, Y)$ for any metric space Y.*
(2) *The set A is a Lipschitz set for $\mathrm{LS}(X)$.*
(3) *The set A has the following properties:*

(a) *For any $\varepsilon > 0$, A intersects only finitely many ε-step territories of X.*

(b) *For any $\varepsilon > 0$ and any ε-step territory E of X, there is a finite constant K_E such that*

$$d_\varepsilon(x, y) \leq K_E d(x, y) \text{ for all } x, y \in A \cap E.$$

(c) *For any $\varepsilon > 0$ and any two distinct ε-step territories E_1 and E_2 of X, there is a finite constant K such that*

$$d(a_1, a_2) \vee d(b_1, b_2) \leq K(d(a_1, b_1) \vee d(a_2, b_2))$$

for all $a_1, a_2 \in A \cap E_1$ and all $b_1, b_2 \in A \cap E_2$.

Proof. The implication (1) $\Longrightarrow$ (2) is trivial.

Assume that condition (2) holds. Suppose that condition (3)(a) fails. There exist $\varepsilon > 0$ and a sequence (x_n) in A so that $x_m \not\sim_\varepsilon x_n$ if $m \neq n$. Set $c_n = nd(x_n, x_1)$ for all n and define $f : X \to \mathbb{R}$ by $f(x) = c_n$ if $x \sim_\varepsilon x_n$ for some n and $f(x) = 0$ otherwise. It is easy to see that $f(x) = f(y)$ if $d(x, y) \leq \varepsilon$. Hence $f \in \mathrm{LS}(X)$. However, $|f(x_n) - f(x_1)| = nd(x_n, x_1)$ for all n. Thus $f_{|(x_n)}$ is not Lipschitz and hence $f_{|A}$ is not Lipschitz, contrary to condition (2).

Next, assume that condition (2) holds but condition (3)(b) fails. There exist $\varepsilon > 0$, an ε-step territory E of X and sequences $(x_n), (y_n)$ in $A \cap E$ so that $x_n \neq y_n$ and

$$d_\varepsilon(x_n, y_n) > nd(x_n, y_n) \text{ for all } n.$$

In particular, $d(x_n, y_n) > \varepsilon$ for all n and hence $d_\varepsilon(x_n, y_n) \to \infty$. Thus at least one of the sequences (x_n) or (y_n) is unbounded in the d_ε-metric. By taking subsequences and switching x_n with y_n if necessary, we may assume that

$$\begin{aligned} d_\varepsilon(x_n, x_1) &\geq \max\{2d_\varepsilon(x_{n-1}, x_1), d_\varepsilon(y_n, x_1)\} \\ \text{and } d_\varepsilon(x_n, y_n) &\geq 2d_\varepsilon(x_{n-1}, y_{n-1}) \end{aligned}$$

for all $n \geq 2$. Let $r_n = \frac{1}{8} d_\varepsilon(x_n, y_n)$. Observe that

$$r_n \leq \frac{1}{8}(d_\varepsilon(x_n, x_1) + d_\varepsilon(y_n, x_1)) \leq \frac{1}{4} d_\varepsilon(x_n, x_1).$$

For $n \geq 2$, define $f_n : E \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} r_n - d_\varepsilon(x_n, x) & \text{if } d_\varepsilon(x_n, x) \leq r_n, \\ 0 & \text{otherwise.} \end{cases}$$

Then each f_n is Lipschitz with Lipschitz constant 1 with respect to the metric d_ε. Suppose that $2 \leq m < n$, $d_\varepsilon(x_m, u) \leq r_m$ and $d_\varepsilon(x_n, v) \leq r_n$. Then

$$\begin{aligned} d_\varepsilon(u, v) &\geq d_\varepsilon(x_n, x_1) - d_\varepsilon(x_m, x_1) - d_\varepsilon(x_m, u) - d_\varepsilon(x_n, v) \\ &\geq d_\varepsilon(x_n, x_1) - d_\varepsilon(x_m, x_1) - r_m - r_n \\ &\geq d_\varepsilon(x_n, x_1) - d_\varepsilon(x_m, x_1) - \frac{1}{4} d_\varepsilon(x_m, x_1) - \frac{1}{4} d_\varepsilon(x_n, x_1) \\ &\geq \frac{1}{8} d_\varepsilon(x_n, x_1) \geq \frac{r_n}{2}. \end{aligned}$$

It follows that the functions $f_n, n \geq 2$, are pairwise disjoint. Thus the function $f = \bigvee_{n \geq 2} f_n$ (pointwise supremum) is well defined (on E). Let u, v be two points in E. If $d_\varepsilon(u, x_m) \leq r_m$ and $d_\varepsilon(v, x_n) \leq r_n$ for some $2 \leq m < n$, then

$$|f(u) - f(v)| \leq |f_m(u)| + |f_n(v)| \leq r_n + r_m \leq 2r_n \leq 4d_\varepsilon(u, v)$$

by the above. Otherwise, there exists $n \geq 2$ such that $f(u) = f_n(u)$ and $f(v) = f_n(v)$. Since f_n has Lipschitz constant 1 with respect to d_ε, $|f(u) -$

$f(v)| \leq d_\varepsilon(u,v)$. This proves that f is Lipschitz with respect to d_ε on E. Extend f by defining it to be 0 on $X\backslash E$. Then f belongs to $\mathrm{LS}(X)$. By condition (2), $f_{|A}$ is Lipschitz with respect to d. Obviously, $f(x_n) = r_n$ for any $n \geq 2$. We claim that $d_\varepsilon(y_n, x_m) > r_m$ for any $m \geq n \geq 2$. Once the claim is established, it follows that

$$|f(y_n)| = \bigvee_{k=2}^{n-1} |f_k(y_n)| \leq \bigvee_{k=2}^{n-1} r_k \leq \frac{r_n}{2}.$$

Therefore,

$$|f(x_n) - f(y_n)| \geq \frac{r_n}{2} = \frac{1}{16} d_\varepsilon(x_n, y_n) > \frac{n}{16} d(x_n, y_n)$$

for all $n \geq 2$, contrary to the Lipschitzness of f with respect to d. To complete the proof, it remains to verify the claim. If $m = n$, then $d_\varepsilon(y_n, x_n) > r_n$ by definition. Suppose that $m > n$. Then

$$\begin{aligned} d_\varepsilon(y_n, x_m) &\geq d_\varepsilon(x_m, x_1) - d_\varepsilon(y_n, x_1) \\ &\geq d_\varepsilon(x_m, x_1) - d_\varepsilon(x_n, x_1) \\ &\geq \frac{1}{2} d_\varepsilon(x_m, x_1) \geq 2r_m > r_m. \end{aligned}$$

Next, we show that (2) $\implies$ 3(c). Let E_1 and E_2 be two distinct ε-step territories of X. Pick $e_1 \in E_1$ and $e_2 \in E_2$. Define $f : X \to \mathbb{R}$ by

$$f(x) = \begin{cases} d(e_1, x) & \text{if } x \in E_1, \\ -d(e_2, x) & \text{if } x \in E_2, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check that $|f(x) - f(y)| \leq d(x,y)$ if $d(x,y) \leq \varepsilon$. Hence $f \in \mathrm{LS}(X)$. By condition (2), $f_{|A}$ is Lipschitz. Let C be a finite constant so that $|f(x) - f(y)| \leq Cd(x,y)$ for all $x, y \in A$. For any $a \in E_1$ and any $b \in E_2$,

$$|f(a) - f(b)| = d(e_1, a) + d(e_2, b) \geq d(e_1, a) \vee d(e_2, b).$$

Suppose that $a_1, a_2 \in A \cap E_1$ and $b_1, b_2 \in A \cap E_2$. Then

$$\begin{aligned} d(a_1, a_2) \vee d(b_1, b_2) &\leq 2[d(e_1, a_1) \vee d(e_1, a_2) \vee d(e_2, b_1) \vee d(e_2, b_2)] \\ &\leq 2(|f(a_1) - f(b_1)| + |f(a_2) - f(b_2)|) \\ &\leq 2C(d(a_1, b_1) + d(a_2, b_2)) \\ &\leq 4C(d(a_1, b_1) \vee d(a_2, b_2)). \end{aligned}$$

Finally, we will show that (3) $\implies$ (1). Let Y be any metric space and let $f \in \mathrm{LS}(X, Y)$. Choose $\varepsilon > 0$ and $C < \infty$ so that $d(f(x), f(y)) \leq Cd(x,y)$ for all $x, y \in X$ with $d(x,y) \leq \varepsilon$. By condition 3(a), the set A meets only finitely many ε-step territories of X, say $E_1, \ldots, E_n$. Note that $f_{|E_i}$ is

Lipschitz with respect to d_ε with Lipschitz constant C for $1 \leq i \leq n$. For each i, choose $e_i \in A \cap E_i$. Set

$$M = 1 \vee \frac{1}{\varepsilon} \max\{d(e_i, e_j) \vee d(f(e_i), f(e_j)) : 1 \leq i \neq j \leq n\}$$

and $L = \bigvee_{i=1}^n K_{E_i}$. Suppose that $x, y \in A \cap E_i$ for some $1 \leq i \leq n$. Using the Lipschitz condition of $f_{|E_i}$ and condition 3(b), we have

$$d(f(x), f(y)) \leq C d_\varepsilon(x, y) \leq C K_{E_i} d(x, y) \leq C L d(x, y).$$

Next, consider the case where $x \in A \cap E_i$ and $y \in A \cap E_j$ with $1 \leq i \neq j \leq n$. Observe that $d(x, y) > \varepsilon$ and hence

$$d(e_i, e_j) \leq \frac{d(e_i, e_j)}{\varepsilon} \cdot d(x, y) \leq M d(x, y).$$

By condition 3(c),

$$d(x, e_i) \vee d(y, e_j) \leq K[(d(x, y) \vee d(e_i, e_j)] \leq K M d(x, y).$$

Thus, using condition 3(b) as well, we have

$$\begin{aligned} d_\varepsilon(x, e_i) \vee d_\varepsilon(y, e_j) &\leq K_{E_i} d(x, e_i) \vee K_{E_j} d(y, e_j) \\ &\leq L K M d(x, y). \end{aligned}$$

Using the Lipschitzness of $f_{|E_i}$ and $f_{|E_j}$ with respect to d_ε, we have

$$\begin{aligned} d(f(x), f(y)) &\leq d(f(x), f(e_i)) + d(f(e_i), f(e_j)) + d(f(e_j), f(y)) \\ &\leq C[d_\varepsilon(x, e_i) + d_\varepsilon(y, e_j)] + \varepsilon M \\ &\leq 2CLKM d(x, y) + M d(x, y) \\ &= M(2CLK + 1) d(x, y). \end{aligned}$$

The two cases above show that $f_{|A}$ is a Lipschitz function. □

A metric space X is said to be *small-determined* if $\mathrm{LS}(X) = \mathrm{Lip}(X)$ [4]. Clearly, X is small-determined if and only if X is a Lipschitz set for $\mathrm{LS}(X)$. Hence, by taking $A = X$ in Theorem 1, one obtains an intrinsic characterization of small-determined metric spaces. An extrinsic characterization of small-determined metric spaces is given in [4, Theorem 6]. A similar intrinsic characterization of small-determined metric spaces was given in [5, Theorem 3.1]. Characterization of small-determined metric spaces solves the problem of comparing $\mathrm{LS}(X)$ and $\mathrm{Lip}(X)$ as spaces of functions. One may also study the comparison of $\mathrm{LS}(X)$ and $\mathrm{Lip}(Y)$ as vector lattices. This is the second main result of the paper. Let X and Y be metric spaces and consider the spaces $\mathrm{LS}(X)$ and $\mathrm{Lip}(Y)$. Any function in $\mathrm{LS}(X)$ has a unique extension to a function on the completion $\widehat{X}$ that is Lipschitz in the small on $\widehat{X}$. A similar statement holds for $\mathrm{Lip}(Y)$. Thus $\mathrm{LS}(X)$ is linearly isomorphic to $\mathrm{LS}(\widehat{X})$ as a unital vector lattice, and $\mathrm{Lip}(Y)$ is linearly isomorphic to $\mathrm{Lip}(\widehat{Y})$ as a unital vector lattice. Therefore, when comparing $\mathrm{LS}(X)$ and $\mathrm{Lip}(Y)$ as vector lattices, we may as well assume that both X and Y are complete metric spaces. From hereon, consider a (not necessarily linear) order isomorphism $T : \mathrm{LS}(X) \to \mathrm{Lip}(Y)$. That is T is a bijection such that $f \geq g$ if and only if $Tf \geq Tg$. Since $\mathrm{LS}(X)$ and $\mathrm{Lip}(Y)$ are vector lattices, in the terminology of [7], they are near vector lattices (see [7, Examples A(a)]. Also, $\mathrm{LS}(X)$ and $\mathrm{Lip}(Y)$ both satisfy (♠) and (♡). (For $\mathrm{Lip}(Y)$, see [7, Examples B(b) and C(c)]. The assertion for $\mathrm{LS}(X)$ can be verified similarly.) Thus, by [7, Theorem 4.5], there exist a homeomorphism $\varphi : X \to Y$ and a function $\Phi : Y \times \mathbb{R} \to \mathbb{R}$ such that for each $y \in Y$, $\Phi(y, \cdot)$ is an increasing homeomorphism from $\mathbb{R}$ onto itself, and that

$$Tf(y) = \Phi(y, f(\varphi^{-1}(y))) \text{ for all } f \in \mathrm{LS}(X) \text{ and all } y \in Y. \tag{1}$$

Lemma 2. *For any $\varepsilon > 0$, X has only finitely many ε-step territories.*

Proof. Suppose on the contrary that there exist $\varepsilon > 0$ and an infinite sequence (x_n) in X so that $x_m \not\sim_\varepsilon x_n$ if $m \neq n$. Represent T as in equation (1). Let $y_n = \varphi(x_n)$ for all n. Choose $c_n \in \mathbb{R}$ so that $\Phi(y_n, c_n) = nd(y_n, y_1)$. Define $f : X \to \mathbb{R}$ by

$$f(x) = \begin{cases} c_n & \text{if } x \sim_\varepsilon x_n, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $f(u) = f(v)$ if $d(u, v) \leq \varepsilon$. Hence $f \in \mathrm{LS}(X)$. Thus $Tf \in \mathrm{Lip}(Y)$. However,

$$Tf(y_n) = \Phi(y_n, f(x_n)) = \Phi(y_n, c_n) = nd(y_n, y_1) \text{ for all } n.$$

In particular,

$$|Tf(y_n) - Tf(y_1)| = nd(y_n, y_1) \text{ for all } n,$$

contradicting the Lipschitzness of Tf. □

Fix a point $y_0 \in Y$. For any $f \in \mathrm{Lip}(Y)$, let

$$L(f) = \sup\{\frac{|f(u) - f(v)|}{d(u, v)} : u, v \in Y, u \neq v\}.$$

Note that $\mathrm{Lip}(Y)$ is a Banach space under the norm

$$\|f\| = |f(y_0)| \vee L(f).$$

$\mathrm{Lip}(Y)$ is important in the general theory of metric spaces, in that for any metric space (Y, d), $y \longmapsto \hat{y} : \mathrm{Lip}(Y) \to \mathbb{R}$ defined by $\hat{y}(f) = f(y)$ is an isometric embedding of Y into the continuous dual of $\mathrm{Lip}(Y)$, leading to an alternative construction of the completion of (Y, d). This is essentially due to E. Michael (although he worked with pointed metric spaces) [6].

For any $N \in \mathbb{N}$, let F_N be the set of all real-valued functions f on X such that $|f(x) - f(y)| \leq Nd(x, y)$ if $d(x, y) \leq \frac{1}{N}$.

Lemma 3. *A real-valued function f on X belongs to* $\mathrm{LS}(X)$ *if and only if there exists $N \in \mathbb{N}$ such that $f \in F_N$.*

Proof. Obviously $F_N \subseteq \mathrm{LS}(X)$. Suppose that $f \in \mathrm{LS}(X)$. There exists $\varepsilon > 0$ and a finite constant K so that $|f(x) - f(y)| \leq Kd(x,y)$ if $d(x,y) \leq \varepsilon$. Choose N so that $N \geq \max\{\frac{1}{\varepsilon}, K\}$. It is easy to check that $f \in F_N$. □

Lemma 4. *For any $N \in \mathbb{N}$, $T(F_N)$ is a closed subset of* $\mathrm{Lip}(Y)$*.*

Proof. Let (f_n) be a sequence in F_N so that (Tf_n) converges to some function g in $\mathrm{Lip}(Y)$. Represent T as in equation (1) above. Take any $x \in X$ and let $y = \varphi(x)$. Since $(Tf_n(y))$ converges to $g(y)$, $(\Phi(y, f_n(x)))$ converges to $g(y)$. As $\Phi(y, \cdot)$ is a homeomorphism on $\mathbb{R}$, $(f_n(x))$ converges to a number $f(x)$ so that $\Phi(y, f(x)) = g(y)$. Suppose that $u, v \in X$ with $d(u,v) \leq \frac{1}{N}$. Then

$$|f(u) - f(v)| = \lim_n |f_n(u) - f_n(v)| \leq Nd(u,v).$$

Thus $f \in F_N$. Since

$$Tf(y) = \Phi(y, f(\varphi^{-1}(y))) = g(y) \text{ for all } y,$$

$g = Tf \in T(F_N)$. □

Theorem 5. *Let X be a complete metric space. Then $\mathrm{LS}(X)$ is* linearly *order isomorphic to* $\mathrm{Lip}(Y)$ *for some complete metric space Y if and only if there exists $N \in \mathbb{N}$ so that X has only finitely many $\frac{1}{N}$-step-territories $E_1, \ldots, E_n$, and that* $\mathrm{LS}(E_k, d) = \mathrm{Lip}(E_k, d_{\frac{1}{N}})$ *for each k.*

Proof. Let $T : \mathrm{LS}(X) \to \mathrm{Lip}(Y)$ be an order isomorphism. By Lemma 3, $\mathrm{Lip}(Y) = T(\mathrm{LS}(X)) = \cup_N T(F_N)$. By Lemma 4, each $T(F_N)$ is a closed set in $\mathrm{Lip}(Y)$. By the Baire Category Theorem, there exists $N \in \mathbb{N}$ so that $T(F_N)$ has nonempty interior in $\mathrm{Lip}(Y)$. By Lemma 2, X has finitely many $\frac{1}{N}$-step-territories $E_1, \ldots, E_n$. We wish to show that $\mathrm{LS}(E_k, d) = \mathrm{Lip}(E_k, d_{\frac{1}{N}})$ for each k. The inclusion "$\supseteq$" is obvious. Let $f \in \mathrm{LS}(E_k, d)$. There exist $r > 0$ and $C < \infty$ such that $|f(u) - f(v)| \leq Cd(u,v)$ if $u, v \in E_k$ and $d(u,v) \leq r$. Set $r' = \min\left\{\frac{1}{N}, r\right\}$ and extend f to a function $\widetilde{f}$ on X by defining $\widetilde{f}(x) = 0$ for all $x \notin E_k$. Then $\left|\widetilde{f}(u) - \widetilde{f}(v)\right| \leq Cd(u,v)$ if $d(u,v) \leq r'$. Thus $\widetilde{f} \in \mathrm{LS}(X)$. Denote the closed unit ball in $\mathrm{Lip}(Y)$ by B. There are $g_0 \in \mathrm{Lip}(Y)$ and $R > 0$ so that $g_0 + RB \subseteq T(F_N)$. Let $T\widetilde{f} = g \in \mathrm{Lip}(Y)$ and choose $t > 0$ so that $t(g - g_0) \in RB$. Since T is linear, we have

$$\begin{aligned} T\widetilde{f} &= \frac{1}{t}[(t-1)g_0 + g_0 + t(g - g_0)] \in \frac{t-1}{t}g_0 + \frac{1}{t}(g_0 + RB) \\ &\subseteq T[\frac{t-1}{t}F_N + \frac{1}{t}F_N]. \end{aligned}$$

Hence $\widetilde{f} \in \frac{t-1}{t} F_N + \frac{1}{t} F_N$. Write $\widetilde{f} = \frac{t-1}{t} h_1 + \frac{1}{t} h_2$ for some $h_1, h_2 \in F_N$. If $u, v \in E_k$ and $d(u,v) \leq \frac{1}{N}$, then

$$\begin{aligned} |f(u) - f(v)| &\leq |\frac{t-1}{t}||h_1(u) - h_1(v)| + \frac{1}{t}|h_2(u) - h_2(v)| \\ &\leq (\frac{|t-1|N}{t} + \frac{N}{t}) d(u,v). \end{aligned}$$

Let $x, y \in E_k$ and $x = x_0, x_1, ..., x_m = y$ be a $\frac{1}{N}$-chain such that $2d_{\frac{1}{N}}(x,y) \geq \sum_{i=1}^m d(x_{i-1}, x_i)$. Then

$$\begin{aligned} |f(x) - f(y)| &\leq \sum_{i=1}^m |f(x_{i-1}) - f(x_i)| \\ &\leq (\frac{|t-1|N}{t} + \frac{N}{t}) \sum_{i=1}^m d(x_{i-1}, x_i) \\ &\leq (\frac{|t-1|N}{t} + \frac{N}{t}) \cdot 2d_{\frac{1}{N}}(x,y). \end{aligned}$$

This proves that $f \in \mathrm{Lip}(E_k, d_{\frac{1}{N}})$.

Conversely, suppose that X has finitely many $\frac{1}{N}$-step territories $E_1, \ldots, E_n$ and that $\mathrm{LS}(E_k, d) = \mathrm{Lip}(E_k, d_{\frac{1}{N}})$ for each k. For each k, choose $e_k \in E_k$. Define a metric ρ on X by

$$\rho(u,v) = \begin{cases} d_{\frac{1}{N}}(u,v) & \text{if } u, v \in E_k \text{ for some } k, \\ d_{\frac{1}{N}}(u, e_j) + 1 + d_{\frac{1}{N}}(e_k, v) & \text{if } u \in E_j,\ v \in E_k \text{ and } j \neq k. \end{cases} \tag{2}$$

Let Y denote the metric space formed by endowing the set X with the metric ρ. To complete the proof, it suffices to show that $\mathrm{LS}(X) = \mathrm{Lip}(Y)$ as sets. Suppose that $f \in \mathrm{LS}(X)$. Define $M = \max\{|f(e_j) - f(e_k)| : 1 \leq j, k \leq n\}$. Since $f_{|E_k} \in \mathrm{LS}(E_k, d) = \mathrm{Lip}(E_k, d_{\frac{1}{N}})$, there exists C_k so that $|f(u) - f(v)| \leq C_k d_{\frac{1}{N}}(u,v)$ if $u, v \in E_k$. Let $C = \max\{C_k :\leq 1 \leq k \leq n\} \vee M$. Then $|f(u) - f(v)| \leq C\rho(u,v)$ if $u, v \in E_k$ for some k. If $u \in E_j$ and $v \in E_k$, where $j \neq k$, then

$$\begin{aligned} |f(u) - f(v)| &\leq |f(u) - f(e_j)| + |f(e_j) - f(e_k)| + |f(e_k) - f(v)| \\ &\leq C d_{\frac{1}{N}}(u, e_j) + M + C d_{\frac{1}{N}}(e_k, v) \\ &\leq C \left(d_{\frac{1}{N}}(u, e_j) + 1 + d_{\frac{1}{N}}(e_k, v) \right) \\ &= C\rho(u,v) \end{aligned}$$

The two estimates above show that $f \in \mathrm{Lip}(Y)$.

Finally, suppose that $f \in \mathrm{Lip}(Y)$. There exists a finite constant C so that $|f(u) - f(v)| \leq C\rho(u,v)$ for all $u, v \in Y$. Consider $u, v \in X$ with $d(u,v) \leq \frac{1}{N}$. Then there exists k such that $u, v \in E_k$. Thus $|f(u) - f(v)| \leq C\rho(u,v) = Cd_{\frac{1}{N}}(u,v) = Cd(u,v)$. This proves that $f \in \mathrm{LS}(X)$, as desired. □

From the proof of Theorem 5, we can obtain some other characterizations of spaces X such that $\mathrm{LS}\,(X)$ is linearly order isomorphic to some $\mathrm{Lip}\,(Y)$. We say that two metrics ρ and d on a space X are *LS equivalent* if the identity map $i:(X,d)\to(X,\rho)$ and its inverse i^{-1} are both LS functions.

Corollary 6. *Let X be a complete metric space. The following are equivalent.*

(a) $\mathrm{LS}(X)$ is linearly order isomorphic to $\mathrm{Lip}(Y)$ for some complete metric space Y.

(b) There exists a metric ρ on X, Lipschitz in the small equivalent to d, such that (X,ρ) is a small-determined metric space.

(c) (X,d) is Lipschitz in the small homeomorphic to some complete small-determined metric space Y.

Proof. $(a) \implies (b)$. Theorem 5 shows that if (a) holds, then there exists $N\in\mathbb{N}$ so that X has only finitely many $\frac{1}{N}$-step-territories $E_1,\dots,E_n$, and that $\mathrm{LS}(E_k,d)=\mathrm{Lip}(E_k,d_{\frac{1}{N}})$ for each k. Define a metric ρ on X as in equation (2). Note that if either $d(u,v)$ or $\rho(u,v)\le\frac{1}{N}$, then $d(u,v)=d_{\frac{1}{N}}(u,v)=\rho(u,v)$. Therefore ρ is LS equivalent to d. In particular, $LS(X,\rho)=LS(X,d)$. Also it follows from the proof of Theorem 5 that $\mathrm{Lip}\,(X,\rho)=\mathrm{LS}\,(X,d)$. Hence (b) holds.
$(b) \implies (c)$ is trivial.
$(c) \implies (a)$. Suppose that $h:X\to Y$ is an LS homeomorphism onto a complete small determined space Y. Then the map $f\mapsto f\circ h^{-1}$ is a linear order isomorphism from $\mathrm{LS}\,(X)$ to $\mathrm{LS}\,(Y)=\mathrm{Lip}\,(Y)$. Therefore, (a) holds. □

Remarks.

(1) Since $d(u,v)=d_{\frac{1}{N}}(u,v)$ if $d(u,v)\le 1/N$, it is easy to see that $\mathrm{LS}(E_k,d)=\mathrm{LS}(E_k,d_{\frac{1}{N}})$. Thus the condition $\mathrm{LS}(E_k,d)=\mathrm{Lip}(E_k,d_{\frac{1}{N}})$ in Theorem 5 is equivalent to the fact that E_k is a Lipschitz set (with respect to the metric $d_{\frac{1}{N}}$) for $\mathrm{LS}(E_k,d_{\frac{1}{N}})$. This condition is characterized by statement (3) in Theorem 1.

(2) We do not know the corresponding result for Theorem 5 if T is merely assumed to be a nonlinear order isomorphism. It should be observed that none of the Lemmas 2 to 4 require the linearity of T.

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