PERSISTENCE OF BANACH LATTICES UNDER NONLINEAR ORDER ISOMORPHISMS

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Abstract. Ordered vector spaces E and F are said to be order isomorphic if there is a (not necessarily linear) bijection $T : E \to F$ such that $x \geq y$ if and only if $Tx \geq Ty$ for all $x, y \in E$. We investigate some situations under which an order isomorphism between two Banach lattices implies the persistence of some linear lattice structure. For instance, it is shown that if a Banach lattice E is order isomorphic to $C(K)$ for some compact Hausdorff space K, then E is (linearly) isomorphic to $C(K)$ as a Banach lattice. Similar results hold for Banach lattices order isomorphic to c_0, and for Banach lattices that contain a closed sublattice order isomorphic to c_0.

Two ordered vector spaces E and F are said to be *order isomorphic* if there is a (not necessarily linear) bijection $T : E \to F$ so that $x \geq y$ if and only if $Tx \geq Ty$ for all $x, y \in E$. In this case, we call T an *order isomorphism*. When E and F are Banach lattices, there is the well studied notion of (vector) lattice isomorphism: E and F are *lattice isomorphic* if there is a linear bijection $T : E \to F$ such that $T|x| = |Tx|$ for all $x \in E$. This is equivalent to the existence of a linear order isomorphism from E onto F. It is well known that a lattice isomorphism T between Banach lattices must also be an isomorphism between the underlying Banach spaces; that is, both T and T^{-1} must be bounded. It is easy to see that, in general, two Banach lattices that are order isomorphic need not be lattice isomorphic. Indeed, for any measure space (Ω, Σ, μ) and any $1 < p < \infty$, the map $f \mapsto |f|^p \operatorname{sgn} f$ is an order isomorphism from $L^p(\Omega, \Sigma, \mu)$ onto $L^1(\Omega, \Sigma, \mu)$. However, $L^p(\Omega, \Sigma, \mu)$ and $L^1(\Omega, \Sigma, \mu)$ are not lattice isomorphic unless they are finite dimensional. In contrast to the situation for L^p spaces, it is shown in this paper that some vector lattice properties pertaining to AM-(or abstract M-) spaces persist under order isomorphisms. For the definition of AM-spaces, as well as for general background with regard to the theory of Banach lattices, we refer the reader to [5, 7]. By the well known Kakutani's representation theorem, a Banach lattice is an AM-space if and only if it is isometrically lattice isomorphic to a closed sublattice of $C(K)$ for some

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compact Hausdorff space K; see, e.g., [4, Theorem 1.b.6]. Our first result is quite simple. If u is a positive element in a Banach lattice E, let E_u be the closed ideal in E generated by u,

$$E_u = \{x \in E : |x| \leq nu \text{ for some } n \in \mathbb{N}\}.$$

u is an *order unit* of E if $E_u = E$. It is a standard fact that if E has an order unit, then E is lattice isomorphic to $C(K)$ for some compact Hausdorff space K; see [7, Proposition II.7.2 and Corollary 1 to Theorem II.7.4].

Theorem 1. *Let E be a Banach lattice. If E is order isomorphic to $C(K)$ for some compact Hausdorff space K, then E is lattice isomorphic to $C(K)$.*

Proof. Let $T : C(K) \to E$ be an order isomorphism. We may assume that $T0 = 0$. For any $n \in \mathbb{N}$, we use the same symbol n denote the constant function on K with value n. Then $C(K)_+ = \cup_n [0, n]$. Hence $E_+ = \cup_n [0, x_n]$, where $x_n = Tn$. By the Baire Category Theorem, there exists n_0 such that $[0, x_{n_0}]$ contains nonempty interior. Thus E_+ has an interior point u. By [5, Corollary 1.2.14], u is an order unit of E. It follows that E is lattice isomorphic to $C(L)$ for some compact Hausdorff space L. In this case, $C(K)$ and $C(L)$ are nonlinearly order isomorphic. By [1, Proposition 3], K and L are homeomorphic. Thus $C(K)$ and $C(L)$ are lattice isomorphic. Since E is lattice isomorphic to $C(L)$, the proof is complete. □

We do not know if a Banach lattice that is order isomorphic to an AM-space must be lattice isomorphic to an AM-space. In this direction, there is a useful characterization of AM-spaces due to Cartwright and Lotz; see [2] and [5, Theorem 2.1.12]. A subset A in an ordered vector space E is *order bounded* if there are $u, v \in E$ such that $u \leq x \leq v$ for all $x \in A$. A sequence (x_n) in a vector lattice is *disjoint* if $|x_m| \wedge |x_n| = 0$ whenever $m \neq n$.

Theorem 2. (Cartwright and Lotz) *A Banach lattice E is lattice isomorphic to an AM-space if and only if every disjoint norm null sequence in E is order bounded in E''.*

With the help of this theorem, we offer a partial solution to the problem raised above. A subspace F of a Banach lattice E is an *(order) ideal* if $y \in F$ for all $y \in E$ such that $|y| \leq |x|$ for some $x \in F$. By [5, Proposition 2.1.9], every closed ideal in $C(K)$ has the form

$$I = \{f \in C(K) : f = 0 \text{ on } K_0\} \text{ for some closed subset } K_0 \text{ of } K.$$

Proposition 3. *Let E be a Banach lattice. If E is order isomorphic to a closed ideal of some space $C(K)$, where $C(K)$ is separable, then E is lattice isomorphic to an AM-space.*

Proof. Let $T : E \to I$ be an order isomorphism, where I is a closed ideal in $C(K)$, with $C(K)$ separable. We may assume that $T0 = 0$. Since $C(K)$ is separable, K is metrizable. Let d be a metric on K generating the given topology. There is a closed set K_0 in K so that I consists of all functions in $C(K)$ that vanish on K_0. By Theorem 2, it suffices to show that every disjoint norm null sequence in E is order bounded in E. Let (x_n) be a disjoint null sequence in E. Define $f_n = T|x_n|$ for all n. Then (f_n) is a disjoint nonnegative sequence in I. If (f_n) is not norm bounded, there is a subsequence (f_{n_k}) such that $\|x_{n_k}\| \leq 1/2^k$ and $\|f_{n_k}\| > k$ for all k. The sum $x = \sum |x_{n_k}|$ converges in E. Clearly $Tx \geq f_{n_k} \geq 0$ for all k. This implies that $\|Tx\| > k$ for all k, which is absurd. Therefore, there exists c_0 such that $c_0 > \|f_n\|$ for all n.

Claim. Let $c_k = \sup\{f_n(t) : d(t, K_0) \leq 1/k, n \in \mathbb{N}\}$. Then (c_k) is a nonincreasing null sequence.

Clearly (c_k) is a nonincreasing sequence. If (c_k) is not a null sequence, there exists $\varepsilon > 0$ such that $c_k > \varepsilon$ for all k. By uniform continuity of f_n, for each n, $\lim_k \sup\{f_n(t) : d(t, K_0) \leq 1/k\} = 0$. Thus, there exist $n_1 < n_2 < \cdots$ and (t_i) in K, $d(t_i, K_0) \to 0$, such that $f_{n_i}(t_i) > \varepsilon$ for all i. By taking a further subsequence if necessary, we may also assume that $\|x_{n_i}\| \leq 1/2^i$ for all i. Now $x = \sum |x_{n_i}|$ converges in E and $Tx \geq f_{n_i}$ for all i. Then $Tx(t_i) \geq f_{n_i}(t_i) > \varepsilon$ for all i. Since $d(t_i, K_0) \to 0$, this contradicts the fact that $Tx \in I$.

By the Claim, there exists a continuous function g on $[0, \infty)$ such that $g(0) = 0$, $g(s) \geq c_k$ if $\frac{1}{k+1} \leq s < \frac{1}{k}$, where we take $1/0 = \infty$. Define $f : K \to \mathbb{R}$ by $f(t) = g(d(t, K_0))$. Then $f \in C(K)$ and $f = 0$ on K_0. Hence $f \in I$. For any n, if $d(t, K_0) = 0$, then $t \in K_0$ and hence $f_n(t) = 0 \leq f(t)$. On the other hand, if $\frac{1}{k+1} \leq d(t, K_0) < \frac{1}{k}$, then $f_n(t) \leq c_k \leq f(t)$. Thus $f \geq f_n$ for all $n \in \mathbb{N}$. Then $T^{-1}f \geq |x_n|$ for all $n \in \mathbb{N}$. Therefore, (x_n) is order bounded in E, as desired. □

Now we can show that the Banach lattice c_0 is stable under nonlinear order isomorphisms.

Theorem 4. *Let E be a Banach lattice. The following are equivalent.*

(a) *E is lattice isomorphic to c_0.*
(b) *E is order isomorphic to c_0.*
(c) *E is order isomorphic to an infinite dimensional closed sublattice of c_0.*

Proof. The implications (a) $\implies$ (b) $\implies$ (c) are immediate. By [6, Corollary 5.3], every infinite dimensional closed sublattice of c_0 is lattice isomorphic to c_0. The implication (c) $\implies$ (b) follows. Now assume that E is order isomorphic to c_0. Let $T : c_0 \to E$ be an order isomorphism such that $T0 = 0$. Denote by (e_n) the unit vector basis of c_0 and let $x_n = Te_n$ for each n. If $m \neq n$, $0 = T(e_m \wedge e_n) = x_m \wedge x_n$. That is, (x_n) is a disjoint positive sequence in E. Also, since $[0, e_n]$ is a totally ordered set, so is $[0, x_n]$. It follows that $[0, x_n] = \{cx_n : 0 \leq c \leq 1\}$.

Claim. For each $n \in \mathbb{N}$ and any $a \geq 0$, there exists $b \geq 0$ such that $T(ae_n) = bx_n$.

Let $a \geq 0$ be given and define $b = \sup\{c \geq 0 : cx_n \leq T(ae_n)\}$. Obviously, the set on the right contains 0 and hence is nonempty. Also $cx_n \leq T(ae_n)$ implies that $|c|\|x_n\| \leq \|T(ae_n)\|$. Since $x_n \neq 0$, it follows that $b < \infty$. There exist $c_k \geq 0$ such that $c_k x_n \leq T(ae_n)$ and $c_k \to b$. Since E_+ is a closed set, $bx_n \leq T(ae_n)$. Let $x = T(ae_n) - bx_n$. Then $x \geq 0$. Thus $T^{-1}x = \sum a_m e_m = \bigvee a_m e_m \geq 0$ in c_0. If $m \neq n$,

$$0 = T(ae_n \wedge e_m) = T(ae_n) \wedge x_m \geq x \wedge x_m \geq 0.$$

Thus $x \wedge x_m = 0$ if $m \neq n$. On the other hand, since $x \wedge x_n \in [0, x_n]$, there exists $0 \leq c \leq 1$ such that $x \wedge x_n = cx_n$. Then

$$T(ae_n) - bx_n = x \geq x \wedge x_n = cx_n$$

and hence $T(ae_n) \geq (b + c)x_n$. By definition of b, $c = 0$. Hence $x \wedge x_n = 0$. Therefore, $T^{-1}x \wedge e_i = T^{-1}(x \wedge x_i) = 0$ for all i. Clearly, this means that $T^{-1}x = 0$ and hence $x = 0$. So we have shown that $T(ae_n) = bx_n$, as desired. This completes the proof of the Claim.

Let x be any positive element in E. Then $T^{-1}x = \bigvee a_n e_n$ for some nonnegative sequence $(a_n) \in c_0$. Thus $x = \bigvee T(a_n e_n)$. By the Claim, $x = \bigvee b_n x_n$ for some nonnegative scalars b_n. If $\bigvee b_n x_n = \bigvee b'_n x_n$, where $b_n, b'_n \geq 0$ and both suprema exist, then using the distributivity of the lattice operations, it is easy to see that $b_n = b'_n$ for all n.

Now we show that for any $x = \bigvee b_n x_n$ as described above, $\lim \|b_n x_n\| = 0$. Otherwise, there exist $\varepsilon > 0$ and an infinite subset I of $\mathbb{N}$ so that $\|b_n x_n\| \geq \varepsilon$ for all $n \in I$. For each $k \in \mathbb{N}$, $T^{-1}(kb_n x_n) = \bigvee_m a_{k,m} e_m$. If $i \neq n$,

$$0 = T^{-1}(kb_n x_n \wedge x_i) = T^{-1}(kb_n x_n) \wedge T^{-1}x_i = (\bigvee_m a_{k,m} e_m) \wedge e_i = (a_{k,i} \wedge 1)e_i.$$

Thus $a_{k,i} = 0$ if $i \neq n$. Hence $T^{-1}(kb_n x_n) = a_{k,n} e_n$. Then $T^{-1}(kx) = \bigvee a_{k,n} e_n$. In particular, $\lim_n a_{k,n} = 0$ for all k. Choose $n_1 < n_2 < \cdots$ in I so that $\lim_k a_{k,n_k} = 0$. We have $z = \bigvee a_{k,n_k} e_{n_k} \in c_0$ and $z \geq T^{-1}(kb_{n_k} x_{n_k})$ for all k. Hence $Tz \geq kb_{n_k} x_{n_k}$ for all k. But then $\|Tz\| \geq \|kb_{n_k} x_{n_k}\| \to \infty$, which is impossible. This proves that $\lim \|b_n x_n\| = 0$.

To recap, we have shown that if $x \in E_+$, then x has a unique representation $x = \bigvee b_n x_n$, where b_n are nonnegative scalars so that $\lim \|b_n x_n\| = 0$. Note that c_0 is a closed ideal in the space $C(\mathbb{N}^*)$, where $\mathbb{N}^*$ is the 1-point compactification of $\mathbb{N}$, and that $C(\mathbb{N}^*)$ is separable. By Proposition 3, E is lattice isomorphic to an AM-space. Consider the linear map $S : c_0 \to E$ given by $S(b_n) = \sum b_n x_n / \|x_n\|$. Note that if $(b_n) \in c_0$, then $\sum b_n x_n / \|x_n\|$ converges in E since E is an AM-space. Since (x_n) is a disjoint sequence, S is an injection. If $x \in E_+$, then $x = \bigvee b_n x_n$, where b_n are nonnegative scalars so that $\lim \|b_n x_n\| = 0$. Thus $S(b_n \|x_n\|) = \sum b_n x_n = \bigvee b_n x_n = x$. Hence the range of S contains E_+. It follows that S is onto. It is clear that $S(b_n) \geq 0$ if $(b_n) \geq 0$. Since S is a bijection as well, S is an order isomorphism. Hence it is a linear order isomorphism and thus a lattice isomorphism. □

In view of Theorems 1 and 4, and the example of L^p spaces mentioned in the introduction, it seems reasonable to ask the following question.

Problem. Suppose that E is a Banach lattice so that any Banach lattice that is order isomorphic to E is (linearly) lattice isomorphic to E. Must E be an AM-space?

We can offer a partial solution to the problem. An element $e \geq 0$ in a Banach lattice is an *atom* if the ideal generated by e is one dimensional. A Banach lattice is *atomic* if there is a maximal orthogonal set consisting of atoms. Let E be an atomic Banach lattice and let $(e_\gamma)_{\gamma\in\Gamma}$ be a maximal orthogonal set consisting of normalized atoms. Any element $x \in E$ has a unique representation

$$x = \bigvee_{\gamma\in\Gamma_1} a_\gamma e_\gamma - \bigvee_{\gamma\in\Gamma_2} a_\gamma e_\gamma, \tag{1}$$

where Γ_1 and Γ_2 are disjoint subsets of Γ and $0 < a_\gamma \in \mathbb{R}$ for all $\gamma \in \Gamma_1 \cup \Gamma_2$. See, e.g., [7, Exercise II.7]. For $1 < p < \infty$, the *p-convexification* of E, denoted by $E^{(p)}$, as defined on p.53 in [4], may be presented as follows. $E^{(p)}$ is the set of all real sequences $(a_\gamma)_{\gamma\in\Gamma}$ such that $\bigvee |a_\gamma|^p e_\gamma \in E$, endowed with the norm $|||(a_\gamma)||| = \| \bigvee |a_\gamma|^p e_\gamma\|^{1/p}$. $E^{(p)}$ is a Banach lattice (in the pointwise order). For each $\gamma \in \Gamma$, let $u_\gamma = (a_\xi)_{\xi\in\Gamma}$ with $a_\xi = 1$ if $\xi = \gamma$ and $a_\xi = 0$ otherwise. Then $(u_\gamma)_{\gamma\in\Gamma}$ is a maximal orthogonal set in $E^{(p)}$ consisting of normalized atoms. The map $T : E^{(p)} \to E$,

$$T(a_\gamma) = \bigvee_{\{\gamma: a_\gamma \geq 0\}} |a_\gamma|^p e_n - \bigvee_{\{\gamma: a_\gamma < 0\}} |a_\gamma|^p e_n$$

is a nonlinear order isomorphism. The norm on a Banach lattice X is said to be *weakly Fatou* [5, Definition 2.4.18] if there is a constant $K < \infty$ so that if $0 \leq x_\tau \uparrow x$, then $\|x\| \leq K \sup_\tau \|x_\tau\|$.

Theorem 5. *Let E be an atomic Banach space and let $(e_\gamma)_{\gamma\in\Gamma}$ be a maximal orthogonal set consisting of normalized atoms. Suppose that any Banach lattice F that is (nonlinearly) order isomorphic to E is (linearly) lattice isomorphic to E. Then the closed sublattice generated by $(e_\gamma)_{\gamma\in\Gamma}$ in E is lattice isomorphic to $c_0(\Gamma)$. Furthermore, if the norm on E is weakly Fatou, then E is lattice isomorphic to a closed sublattice of $\ell^\infty(\Gamma)$.*

Proof. Let F be the 2-convexification of E and let (u_γ) be the maximal orthogonal set of normalized atoms in F as described above.. Since F is order isomorphic to E, it is lattice isomorphic to E by the assumption. Let $T : E \to F$ be a lattice isomorphism. For each γ, Te_γ is a nonzero positive element in F and $[0, Te_\gamma] = T[0, e_\gamma]$ lies within a 1-dimensional subspace. Hence there exist $\pi(\gamma) \in \Gamma$ and $c_\gamma > 0$ such that $Te_\gamma = c_\gamma u_{\pi(\gamma)}$. Since T is a lattice isomorphism, $\pi : \Gamma \to \Gamma$ is a permutation on Γ and

$0 < \inf c_\gamma \leq \sup c_\gamma < \infty$. For any finite subset I of Γ, we have

$$\frac{1}{\sup c_\gamma}\|\bigvee_{\gamma\in I} e_\gamma\| \leq \|\bigvee_{\gamma\in I}\frac{1}{c_\gamma}e_\gamma\| = \|T^{-1}\bigvee_{\gamma\in I} u_{\pi(\gamma)}\|$$
$$\leq \|T^{-1}\|\cdot|||\bigvee_{\gamma\in I} u_{\pi(\gamma)}||| = \|T^{-1}\|\|\bigvee_{\gamma\in I} e_{\pi(\gamma)}\|^{1/2}.$$

Let $C = \sup c_\gamma\|T^{-1}\|$. For any $m \in \mathbb{N}$, let

$$\mu_m = \sup\{\|\bigvee_{n\in I} e_n\| : \#I = m\}.$$

Clearly, $\mu_m < \infty$. Let I be such that $\#I = m$ and $\|\bigvee_{n\in I} e_n\| \geq \mu_m/2$. Then

$$\mu_m \geq \|\bigvee_{\gamma\in\pi(I)} e_\gamma\| \geq \frac{1}{C^2}\|\bigvee_{\gamma\in I} e_\gamma\|^2 \geq \frac{\mu_m^2}{4C^2}.$$

Therefore, $\mu_m \leq 4C^2$.

Let G be the closed sublattice of E generated by $(e_\gamma)_{\gamma\in\Gamma}$. Since $(e_\gamma)_{\gamma\in\Gamma}$ is a disjoint set, G is the same as the closed subspace generated by $(e_\gamma)_{\gamma\in\Gamma}$. Clearly, $\sum a_\gamma e_\gamma \in G$ implies that $(a_\gamma) \in c_0(\Gamma)$. Conversely, suppose that $(a_\gamma) \in c_0(\Gamma)$. For any $\varepsilon > 0$, there exists a finite subset I of Γ such that $|a_\gamma| \leq \varepsilon$ for all $\gamma \notin I$. If J is a finite subset of Γ disjoint from I, then

$$\|\sum_{\gamma\in J} a_\gamma e_\gamma\| \leq \max_{\gamma\in J}|a_\gamma|\|\sum_{\gamma\in J} e_\gamma\| \leq \varepsilon\cdot 4C^2.$$

Thus, $\sum a_\gamma e_\gamma$ converges in G if $(a_\gamma) \in c_0(\Gamma)$. It is now clear that the map $S : c_0(\Gamma) \to G$ defined by $S(a_\gamma) = \sum a_\gamma e_\gamma$ is a lattice isomorphism.

Finally, suppose that the norm on E is weakly Fatou with constant K. By the discussion preceding the theorem, each $x \in E$ has a unique representation (1). Clearly, for $\gamma \in \Gamma_1 \cup \Gamma_2$, $|a_\gamma| = \|a_\gamma e_\gamma\| \leq \|x\|$. Define

$$x(\gamma) = \begin{cases} a_\gamma & \text{if } \gamma \in \Gamma_1, \\ -a_\gamma & \text{if } \gamma \in \Gamma_2, \\ 0 & \text{otherwise.} \end{cases}$$

Then $(x(\gamma)) \in \ell^\infty(\Gamma)$ and $\|(x(\gamma))\|_\infty \leq \|x\|$. On the other hand $\bigvee_{\gamma\in I}|x(\gamma)|e_\gamma \uparrow |x|$, where I runs through the directed set of all finite subsets of Γ. By assumption,

$$\|x\| = \||x|\| \leq K\sup_I\|\bigvee_{\gamma\in I}|x(\gamma)|e_\gamma\|$$
$$= K\sup_I\|S(|x(\gamma)|)_{\gamma\in I}\| \leq K\|S\|\|(x(\gamma))\|_\infty.$$

It is now clear that the map $R : E \to \ell^\infty(\Gamma)$ given by $Rx = (x(\gamma))$ is a lattice isomorphism from E onto a closed sublattice of $\ell^\infty(\Gamma)$. □

Our final result shows that containment of a closed sublattice isomorphic to c_0 is also a stable property under order isomorphisms. This holds in fact in the category of quasi-Banach lattices. Let E be a real or complex vector space. A *quasi-norm* on E is a functional $\|\cdot\|$ on E such that

(a) $\|x\| > 0$ if $x \neq 0$,
(b) $\|ax\| = |a|\|x\|$ for any scalar a and any $x \in E$,
(c) There is a constant $C < \infty$ such that $\|x+y\| \leq C(\|x\| \vee \|y\|)$ for all $x, y \in E$.

A quasi-norm on E generates a Hausdorff linear topology where the sets $\{x : \|x\| < 1/n\}$ form a neighborhood basis at 0. If this topology is completely metrizable, then we say that the quasi-norm is *complete* and that E is a *quasi-Banach space*. A *quasi-Banach lattice* is a real vector lattice equipped with a complete quasi-norm $\|\cdot\|$ such that $|x| \leq |y|$ implies $\|x\| \leq \|y\|$ for all $x, y \in E$. Refer to [3] for more information regarding quasi-Banach spaces and quasi-Banach lattices. Given a quasi-norm $\|\cdot\|$ with associated constant C, it is evident that

$$\|\sum_{k=1}^{n} x_k\| \leq \max_{1\leq k\leq n-1} C^k\|x_k\| \vee C^{n-1}\|x_n\|.$$

It follows that if (x_k) is a sequence in a quasi-Banach space with $\lim \|x_k\| = 0$, then there is a subsequence (x_{k_i}) such that $\sum x_{k_i}$ converges. It is easy to see that the positive cone $\{x : x \geq 0\}$ is a closed set in a quasi-Banach lattice; equivalently, the limit of any positive sequence is positive. Consider the following statements.

Theorem 6. *Let E and F be order isomorphic quasi-Banach lattices. If E contains a closed sublattice (nonlinearly) order isomorphic to c_0, then F contains a closed sublattice linearly lattice and topologically isomorphic to c_0.*

Theorem 7. *Let E and F be order isomorphic quasi-Banach lattices. If E contains a closed sublattice* linearly *order isomorphic to c_0, then F contains a closed sublattice linearly lattice and topologically isomorphic to c_0.*

Evidently Theorem 6 is stronger than Theorem 7. But, in fact, the two results are equivalent. Indeed, assume that Theorem 7 holds. If G is a quasi-Banach lattice order isomorphic to c_0, then, taking E to be c_0 and F to be G in Theorem 7, one concludes that G contains a closed sublattice linearly order isomorphic to c_0. Thus any E that satisfies the hypothesis of Theorem 6 also fulfills the condition of Theorem 7.

We now proceed to prove Theorem 7 (and hence also Theorem 6). First observe that in order to produce a closed sublattice of F linearly order and topologically isomorphic to c_0, it suffices to obtain a disjoint sequence (y_i) in F such that $\inf \|y_i\| > 0$ and $\sup_j \|\sum_{i=1}^{j} y_i\| < \infty$.

Lemma 8. *Let G be a quasi-Banach lattice and let $S : c_0 \to G$ be a (linear) lattice isomorphism. Denote by (e_k) the unit vector basis of c_0. Then* $\inf \|Se_k\| > 0$.

Proof. Otherwise, by the observation preceding Theorem 6, there is a subsequence (e_{k_i}) such that $x = \sum Se_{k_i}$ converges in G. Since the positive cone of G is closed, $x \geq \sum_{i=1}^m Se_{k_i} = S(\sum_{i=1}^m e_{k_i})$ for all m. Then $S^{-1}x \geq \sum_{i=1}^m e_{k_i}$ for all m, which is clearly absurd. □

Lemma 9. *Let E and F be quasi-Banach lattices and let $T : E \to F$ be an order isomorphism such that $T0 = 0$. If (x_k) is a disjoint sequence in E_+ with* $\inf \|x_k\| > 0$*, then there exists $N \in \mathbb{N}$ such that* $\limsup \|T(Nx_k)\| > 0$.

Proof. Otherwise, there is a subsequence (x_{k_i}) such that $\lim \|T(ix_{k_i})\| = 0$. By using a further subsequence if necessary, we may assume that $y = \sum T(ix_{k_i})$ converges in F. Since T is an order isomorphism and $T0 = 0$, $T(ix_{k_i}) \geq 0$ for all i. Thus $y \geq T(ix_{k_i}) \geq 0$ for all i. Hence $T^{-1}y \geq ix_{k_i} \geq 0$ for all i. Therefore, $\|T^{-1}y\| \geq i\|x_{k_i}\| \geq i \inf_k \|x_k\|$ for all i, which is impossible. □

Proof of Theorem 7. Let G be a closed sublattice of E and let $S : c_0 \to G$ and $T : E \to F$ be order isomorphisms, where S is linear and, without loss of generality, $T0 = 0$. Denote the unit vector basis of c_0 by (e_k). By Lemma 8, $\inf \|Se_k\| > 0$. Let $x_k = Se_k$. Since (x_k) is a disjoint sequence in E_+, by Lemma 9, there exists $N \in \mathbb{N}$ and an infinite subset I of $\mathbb{N}$ so that $\inf_{k\in I} \|T(Nx_k)\| > 0$.

Assume that there exists $\eta > 0$ such that $\inf_{k\in I} \|T(\eta x_k)\| = 0$. There is an increasing sequence (k_i) in I such that $y = \sum T(\eta x_{k_i})$ converges in F. Then $y \geq T(\eta x_{k_i})$ and hence $T^{-1}y \geq \eta x_{k_i}$ for all i. Thus $x = (N/\eta)T^{-1}y \geq Nx_{k_i}$ and so $Tx \geq T(Nx_{k_i})$ for all i. Let $y_i = T(Nx_{k_i})$ for all i. Then (y_i) is a disjoint sequence in F such that $\inf \|y_i\| > 0$. Furthermore, $0 \leq \sum_{i=1}^j y_i = \bigvee_{i=1}^j y_i \leq Tx$ for all j. Hence $\|\sum_{i=1}^j y_i\| \leq \|Tx\|$ for all j. By the remark preceding Lemma 8, F has a closed sublattice linearly order and topologically isomorphic to c_0.

Finally, suppose that $\inf_{k\in I} \|T(\eta x_k)\| > 0$ for all $\eta > 0$. Let (k_i) be an increasing sequence in I. We claim that there exists $\varepsilon > 0$ such that $\sup_j \|T(\varepsilon \sum_{i=1}^j x_{k_i})\| < \infty$. Otherwise, there is an increasing sequence (j_m) such that $\|T(2^{-m} \sum_{i=1}^{j_m} x_{k_i})\| > m$ for all m. The element

$$u = \sum_{m=1}^{\infty} 2^{-m} \sum_{i=1}^{j_m} e_{k_i}$$

belongs to c_0 and majorizes $2^{-m} \sum_{i=1}^{j_m} e_{k_i}$ for each m. Since S is linear and order preserving, $x = Su \geq 2^{-m} \sum_{i=1}^{j_m} x_{k_i}$ and thus $Tx \geq T(2^{-m} \sum_{i=1}^{j_m} x_{k_i}) \geq 0$ for all m. But then $\|Tx\| > m$ for all m, reaching a contradiction. Hence the claim is verified. Let $y_i = T(\varepsilon x_{k_i})$. Then (y_i) is a disjoint sequence and

$\inf \|y_i\| > 0$. For any j,

$$0 \leq \sum_{i=1}^{j} y_i = \bigvee_{i=1}^{j} y_i = \bigvee_{i=1}^{j} T(\varepsilon x_{k_i}) = T(\bigvee_{i=1}^{j} \varepsilon x_{k_i}) = T(\sum_{i=1}^{j} \varepsilon x_{k_i}).$$

Therefore,

$$\sup_j \| \sum_{i=1}^{j} y_i \| \leq \sup_j \|T(\sum_{i=1}^{j} \varepsilon x_{k_i})\| < \infty.$$

Again, by the remark preceding Lemma 8, we conclude that F has a closed sublattice linearly order and topologically isomorphic to c_0. □

Remark. If E is a Banach lattice, then E does not contain a closed sublattice lattice isomorphic to c_0 if and only if E is weakly sequentially complete. Thus Theorem 6 shows that the topological property of weak sequential completeness is preserved under nonlinear order isomorphisms between Banach lattices.

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