Quasi-synchronization of heterogeneous networks with a generalized Markovian topology and event-triggered communication

Xinghua Liu, *Member, IEEE,* Wee Peng Tay, *Senior Member, IEEE,*
Zhi-Wei Liu, *Member, IEEE,* Gaoxi Xiao, *Member, IEEE,*

***Abstract*—We consider the quasi-synchronization problem of a continuous time generalized Markovian switching heterogeneous network with time-varying connectivity, using pinned nodes that are event-triggered to reduce the frequency of controller updates and inter-node communications. We propose a pinning strategy algorithm to determine how many and which nodes should be pinned in the network. Based on the assumption that a network has limited control efficiency, we derive a criterion for stability, which relates pinning feedback gains, the coupling strength and the inner coupling matrix. By utilizing the stochastic Lyapunov stability analysis, we obtain sufficient conditions for exponential quasi-synchronization under our stochastic event-triggering mechanism, and a bound for the quasi-synchronization error. Numerical simulations are conducted to verify the effectiveness of the proposed control strategy.**

***Index Terms*—Quasi-synchronization, heterogeneous network, generalized Markovian topology, event-triggered control.**

I. Introduction

Complex dynamical networks are ubiquitous in nature and in the modern world. A large variety of social and biological systems, critical infrastructural and communications systems, can be modeled and analyzed as complex networks [1]–[3]. In many of these networks, the state of every network node needs to be synchronized in order to drive the entire system to a certain desired global state. For example, a government agency may want to propagate important emergency coordination information to everyone in an online social network. In a power grid, every active component that generates power should oscillate at the same frequency (50 Hz in Europe; 60 Hz in the US); otherwise, the system may lose its stability and power imbalance may occur.

Synchronization of complex networks has attracted much attention, see [4]–[7] and the references therein for more details. When the network cannot be stabilized or synchronized by itself, one may apply control inputs to drive the network to synchronization. In practice, driving the dynamics of a network from any initial state to a final desired state has many applications in different fields. In the literature, many control strategies have been proposed to enable the network to be stabilized. These include pinning control [8]–[11], fuzzy control [12], impulsive control [13], and fixed-time control [14], etc. As it is generally rather expensive, and not necessary to impose controllers on all network nodes, pinning control stands out as a good choice since it can be realized by controlling only a subset of network nodes.

While most of the literature [3]–[13] have focused on synchronization of identical nodes in networks, our paper deals with synchronization in switching heterogeneous networks, which widely exist in real-life applications. For example, local area networks (LANs) that connect Microsoft Windows and Linux based personal computers with Apple Macintosh computers are heterogeneous [15]; in multiple robot manipulators with Lagrangian dynamics, each manipulator has a different inertia matrix due to distinct structures [16]. The existence of heterogeneity may result in possible loss of synchronization and makes synchronization in heterogeneous networks more difficult and complicated. To the best of our knowledge, there have been few satisfactory results on synchronization of heterogeneous networks [17]–[22], not to mention the random switching heterogeneous networks.

By assuming either that the non-identical nodes have a common equilibrium, or individual node dynamics tends to become the same, complete synchronization can be achieved [17]–[19], i.e., $\lim_{t\to\infty} \|x_i(t) - s(t)\| = 0$, where $x_i(t)$ denotes the state of the i-th node and $s(t)$ is a certain reference trajectory. However, in reality and for many practical cases, mismatched parameters and other external factors always imply that $x_i(t) - s(t)$ cannot approach zero with time. The references [20]–[23] hence reported quasi-synchronization among the non-identical nodes, which means that the defining limit of synchronization is bounded within a region around zero, i.e., $\lim_{t\to\infty} \|x_i(t) - s(t)\| \le \varpi$, where $\varpi > 0$. It is also called bounded synchronization in some cases. Recently, [24] investigated the problem of quasi-synchronization for the time-invariant heterogenous networks by applying distributed impulsive control. The paper [25] derived several quasi-synchronization conditions for the time-varying switching heterogeneous networks. However, how to design the quasi-

Xinghua Liu is with the Department of Electrical Engineering, Xi'an University of Technology, Xi'an 710048, China, and also with the School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore. e-mail: liuxh@xaut.edu.cn

Wee Peng Tay and Gaoxi Xiao are with the School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore. e-mail: wptay@ntu.edu.sg, egxxiao@ntu.edu.sg

Zhi-Wei Liu (Corresponding author) is with Huazhong University of Science and Technology, Wuhan, China. e-mail: zwliu@hust.edu.cn

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synchronization condition and optimize the error bound of quasi-synchronization for random switching heterogeneous networks are open problems.

A complex dynamical network (homogeneous or heterogeneous) is a distributed system. Each agent only interacts with neighboring agents. The topology graph is introduced merely to model the interaction among nodes or agents in the whole network. The changes of network topology can be interpreted as changes in interactions among nodes. Investigators typically model randomly switching network topologies as Markov processes. For example, in a marine oil spill monitoring system deployed over an ocean area, the topology of the sensor network may switch, governed by a generalized Markov process [26]. A comprehensive quasi-synchronization analysis of a generalized Markovian switching heterogeneous network is in demand, which provides the motivation of our current study.

With the rapid development of data communication technologies and high performance computing, event-based control algorithms have been proposed to reduce information exchange and computation loads in networked systems while still maintaining satisfactory control performance [27]–[29]. Event-triggered control is particularly suitable for complex networks with limited resources since it effectively reduces the communication cost by executing the control task only when an external event that violates a predetermined condition on the output measurement occurs. An example of such a control strategy is to trigger a node to transmit its local state to its neighbors only when the estimated local node state error exceeds a certain specific threshold value [30]. In some recent publications [31]–[33], the authors investigated event-triggered algorithms for pinning control of complex networks. In [31], pinning exponential synchronization of complex networks via event-triggered communication with combinational measurements was studied. In [32], sufficient conditions are derived and exponential convergence of a global normed error function is proven for the synchronization of time-varying switching networks with uncoupled node dynamics. The paper [33] employed an event-triggered strategy to achieve good stability in coupled dynamical systems with Markovian switching couplings and pinned node set. However, methods for developing an event-triggered strategy to synchronize heterogeneous networks with generalized Markovian switching topologies under pinning control, to the best of our knowledge, have not been investigated.

In this paper, we consider the quasi-synchronization problem of heterogeneous networks with generalized Markovian switching topologies and event-triggered communication. The main contributions of this paper can be outlined as follows:

1) In contrast to the majority of recent publications on synchronization and consensus of identical networked systems, this paper investigates stochastic heterogeneous dynamic networks, which are widely applicable in real-life situations. A heterogeneous network is established to model nonidentical nodes and data sampling within a unified framework where the network topology is governed by a generalized Markov process.
2) To solve the challenging problem of pinning heterogeneous networks with generalized Markovian switching topologies, an algorithm (cf. Algorithm 1) is proposed to identify the number and locations of the network nodes that need to be pinned in a heterogeneous network. Taking into account the restriction of control efficiency, a criterion (cf. Corollary 1) is proposed to show the relationship between pinning feedback gains, the coupling strength and the inner coupling matrix.
3) By utilizing multiple Lyapunov functionals, with the help of stochastic Lyapunov-Krasovskii stability analysis and matrix inequalities technique, two sufficient criteria (cf. Theorem 1 and Corollary 2) are derived such that the heterogeneous networks with generalized Markovian switching topologies can achieve quasi-synchronization exponentially under our stochastic event-triggering mechanism. We also provide an explicit quasi-synchronization error bound.

The rest of this paper is organized as follows. In Section II, we present our problem statement and some technical preliminaries. Our main theoretic framework and proofs are presented in Section III. Numerical examples and simulations are presented in Section IV to verify the proposed results. Finally, Section V concludes the paper.

Notation: The following notations are used throughout this paper. Let $\otimes$ be the Kronecker product. $\mathbb{N}$ denotes the set of non-negative integers, and $\mathbb{R}^+$ is the set of non-negative real numbers. We use $\mathbb{R}^n$ to denote the n dimensional Euclidean space and $\mathbb{R}^{m\times n}$ the set of all $m\times n$ matrices. $F(\theta^-)$ stands for the left limit of a function $F(\theta)$ at θ. We let I be the identity matrix with the proper dimensions. Let $\|x\|$ and $\|A\|$ be the Euclidean norm of a vector x and a matrix A, respectively. The superscripts ^{T} and $^{-1}$ denote matrix transposition and matrix inverse, respectively. Let $\mathrm{He}\{A\} = A + A^T$ be a symmetric matrix. For a symmetric block matrix, we use $\star$ to denote the terms due to symmetry. $X \prec Y$ ($X \succ Y$), where X and Y are both symmetric matrices, means that $X - Y$ is negative (positive) definite. $C([-h_m, 0], \mathbb{R}^{Nn})$ denotes the family of continuous function f from $[-h_m, 0]$ to $\mathbb{R}^{Nn}$ with the norm $|f| = \sup_{-h_m\le\mu\le 0}\|f(\mu)\|$. Finally, $\mathbb{P}$ and $\mathbb{E}$ denote the probability measure and mathematical expectation operator, respectively, of an underlying probability space, which will be clear from the context.

II. Problem Formulation

A. Graph notations

Let $\mathcal{G} = (\mathcal{V}, E, A)$ be a weighted digraph of order N with the set of nodes $\mathcal{V} = \{1, 2, \cdots, N\}$, set of directed edges $E \subset \mathcal{V}\times\mathcal{V}$, and a weighted adjacency matrix $A = [a_{ij}]_{N\times N}$. A directed graph $\mathcal{G}$ contains a directed spanning tree if there exists a node r, called a root, such that there exists a directed path from this node to every other node q, i.e., node q is reachable from root r. An edge in the graph $\mathcal{G}$ is denoted by (i, j) where (i, j) means that node j can receive information from node i. The neighboring set of node i is denoted by $N_i = \{j \in \mathcal{V} : (j, i) \in E\}$. A path from node j to node i is a sequence of edges, (j, i_1), (i_1, i_2), $\cdots$, (i_p, i), with distinct nodes i_k, $k = 1, 2, \cdots, p$. The elements of the adjacency matrix $A = [a_{ij}]_{N\times N}$ are defined as follows: the weight $a_{ij} > 0$ if and only if $(j, i) \in E$, and $a_{ij} = 0$ otherwise. Moreover, we assume that the graph contains no self-loops, i.e., $a_{ii} = 0$ for all $i = 1, 2, \cdots, N$. The Laplacian matrix $L = [l_{ij}]_{N\times N}$ associated with the adjacency matrix A is defined by $l_{ii} = \sum_{k=1,k\neq i}^{N} a_{ik}$, and $l_{ij} = -a_{ij}$ for $i \neq j$.

B. Problem formulation

Consider a heterogeneous complex network with a generalized Markovian switching topology. Node i has the following state equation:

$$\dot{x}_i(t) = B_i x_i(t) + f(x_i(t), t) + c\sum_{j=1}^{N} a_{ij}(r(t))\Gamma(x_j(t) - x_i(t)),\quad i = 1, 2, \cdots, N, \tag{1}$$

where $x_i(t) \in \mathbb{R}^n$ is its state, $f(x_i(t), t)$ is a continuous vector-valued function, and $r(t)$ is a discrete-state stochastic process taking values in a finite set $\mathbb{S} = \{1, 2, \cdots, M\}$, which will be specified later. The matrices $B_i, i = 1, 2, \cdots, N$ are constant matrices of appropriate dimensions. The positive constant c is the coupling strength, and $0 \prec \Gamma = \mathrm{diag}\{\alpha_1, \alpha_2, \cdots, \alpha_n\} \in \mathbb{R}^{n\times n}$ denotes an inner-coupling matrix of the heterogeneous network. The matrix $A(r(t)) = [a_{ij}(r(t))]_{i,j=1}^{N} \in \mathbb{R}^{N\times N}$ represents the outer coupling configuration with mode switching. We consider the switching complex network in an induced graph $\mathcal{G}(r(t)) = (\mathcal{V}, E(r(t)), A(r(t)))$: for each pair of nodes $i \neq j$, $a_{ij}(r(t)) > 0$ if and only if $(j, i) \in E(r(t))$ at time t; otherwise $a_{ij}(r(t)) = 0$.

In this paper, consider a complete filtered probability space $\{\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P}\}$, where Ω is the sample space, $\mathscr{F}$ is a σ-algebra of events, $\mathbb{P}$ is a probability measure defined on $\mathscr{F}$, and $\{\mathscr{F}_t\}_{t\geq 0}$ is a filtration satisfying the usual conditions (i.e., it is increasing and right continuous while the σ-algebra $\mathscr{F}_0$ contains all $\mathbb{P}$-null sets from $\mathscr{F}$). Let $\mathbb{S} = \{1, 2, \cdots, M\}$ be a finite state space. On the complete filtered probability space $\{\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P}\}$, $r(t)$ is a stochastic process with discrete states in a finite set $\mathbb{S}$ and the transition probabilities are defined as $q_{uv}(t, s) \triangleq \mathbb{P}\left(r(t+s) = v \,|\, r(t) = u\right)$.

Under the continuity condition $\lim_{s\to 0^+} q_{uv}(t, s) = \delta_{uv}$, where δ_{uv} is the Kronecker delta function, we have that

$$\lim_{s\to 0^+} \frac{1 - q_{uu}(t, s)}{s} = -\pi_{uu} < \infty,\ \lim_{s\to 0^+} \frac{q_{uv}(t, s)}{s} = \pi_{uv} < \infty,$$

where $\pi_{uv} \geq 0$, $u, v \in \mathbb{S}, u \neq v$, is the transition rate from mode u to v and $\pi_{uu} = -\sum_{v=1,v\neq u}^{N} \pi_{uv}$.

We suppose that observations are made at the sampling instants T_k, $k \geq 1$. Then we have

$$\begin{aligned}&\mathbb{P}\left(r(T_k) = v | r(T_k^-) = u\right)\\ &= q_{uv}(T_k^-, 0^+) = \begin{cases}\pi_{uv}, & u \neq v\\ 1 + \pi_{uu}, & u = v\end{cases}\end{aligned} \tag{2}$$

where T_k^- stands for the left limit of T_k.

Remark 1. *It is worth mentioning that the stochastic process $r(t)$ in this paper is a direct generalization of the traditional Markov process [34]. If we consider a process that is only defined at the sampling instants, from (2) it can be seen that the process is a standard discrete-time Markov process. If we consider $t \in [T_k, T_{k+1})$, the switching rate π_{uv} is defined over the whole time horizon, and thus becomes identical with the definition for the standard continuous-time Markov process.*

Remark 2. *The probabilistic behavior of the network topology has indicated that the controller for the heterogeneous network will be switched from the instants T_k^- to T_k, and remains unchanged over each sampling interval. In contrast to the previous synchronization strategy of switching networks [35]–[37], our strategy may be easier to be implemented since the control only needs to be updated at switching instants. As we shall discuss in Section II-C, the event-triggering mechanism with stochastic sampling significantly decreases the amount of communications between different nodes in a switching heterogeneous network.*

Remark 3. *Let $y_i(t)$ be the output signal of the i-th node, which has a state-space equation described by* (1). *A typical output feedback control strategy can be expressed as $u_i(t) = \sum_{j=1}^{N} a_{ij}(r(t))\Gamma(y_j(t) - y_i(t))$. Yet it requires careful studies to incorporate the event-triggering mechanism into the output feedback control and to design the effective event detector. However, in this paper we adopt the pinning control design of state feedback, then propose an event-triggered and pinning strategy to effectively reduce the frequency of controller updates and inter-node communications.*

Remark 4. *The generalized Markovian switching heterogeneous network is said to achieve output synchronization asymptotically if $\lim_{t\to\infty} \|y_i(t) - y_j(t)\| = 0$, for $i, j = 1, 2, \cdots, N$. It should be noticed that output synchronization of the generalized Markovian switching heterogeneous network depends on non-identical dynamics of the nodes, and the structural topology governed by the stochastic process $r(t)$, which impose the main obstacles for output synchronization. To the best of our knowledge, there are no existing results on event-triggered output synchronization problem for the generalized Markovian switching heterogeneous networks, but the output synchronization of homogeneous or heterogeneous networks has been well investigated in [38]–[40]. However, in this paper we focus on the event-triggered quasi-synchronization problem of* (1), *and the event-triggered output synchronization problem will be studied in our future work.*

In this paper, we define a virtual leader node labeled as $N+1$. By introducing directed edges in the form of $(N+1, i)$, $i \in \mathcal{V}$, to $\mathcal{G} = (\mathcal{V}, E, A)$, one obtains a graph $\tilde{\mathcal{G}} = (\tilde{\mathcal{V}}, \tilde{E}, \tilde{A})$, known as the augmented graph of $\mathcal{G} = (\mathcal{V}, E, A)$. We assume that the dynamics of node $N + 1$ is given by

$$\dot{s}(t) = Bs(t) + f(s(t), t). \tag{3}$$

where $s(t)$ may be an equilibrium point, a periodic orbit, or even a chaotic orbit. Throughout this paper, we assume that $s(t)$ is bounded, i.e., for any initial condition $s(0)$, there exist $T(s(0))$ and $\delta > 0$, such that $\|s(t)\| \leq \delta$, $\forall t > T(s(0))$.

Assumption 1. *The augmented graph $\tilde{\mathcal{G}} = (\tilde{\mathcal{V}}, \tilde{E}, \tilde{A})$ contains a directed spanning tree with the virtual node $N+1$ being the root.*

Since the network (1) and the virtual leader (3) have individual dynamic properties, such a heterogeneous network cannot achieve synchronization from the traditional point of view [21]. We define a more general synchronization concept, called quasi-synchronization, as follows.

Definition 1. *[24] The heterogeneous dynamic network (1) is said to achieve quasi-synchronization with an error bound $\varpi > 0$, if there exists a compact set $\mathcal{M}$ such that, for any $x_i(0), s(0) \in \mathbb{R}^n$, the error signal $e_i(t) = x_i(t) - s(t)$ converges into the set $\mathcal{M} = \{e_i(t) \in \mathbb{R}^n : \|e_i(t)\| \leq \varpi\}$ as $t \to \infty$.*

Throughout this paper, suppose that the event-triggered instants for the node i are $t_0^i, t_1^i, \cdots$, which will be determined by the proposed event detector (cf. Section II-C). Incorporating the event-triggering mechanism in (1), we design an event-triggered and pinning strategy as follows

$$\begin{aligned}\dot{x}_i(t)=&B_i x_i(t)+f(x_i(t),t)+ca_{i0}(r(t))\Gamma(s(t_k^i)-x_i(t_k^i))\\&+c\sum_{j=1}^{N}a_{ij}(r(t))\Gamma(x_j(t_k^i)-x_i(t_k^i)),\end{aligned}\tag{4}$$

where $k = 0, 1, 2, \cdots$, $i = 1, 2, \cdots, N$. Without loss of generality, suppose that $t_0^i = 0$ for $i = 1, 2, \cdots, N$. Let $\mathcal{V}_{\text{pin}}(r(t)) = \{i_1(r(t)), \cdots, i_l(r(t))\}$ be the set of pinned nodes. If $i \in \mathcal{V}_{\text{pin}}(r(t))$, $a_{i0}(r(t)) > 0$; otherwise $a_{i0}(r(t)) = 0$. Furthermore, we define $A_0(r(t)) = \text{diag}\{a_{10}(r(t)), \cdots, a_{N0}(r(t))\}$.

In the following, we present some assumptions and lemmas, which will be used to derive our main results.

Assumption 2. *The continuous vector-valued function $f(\cdot) = [f_1^T(\cdot), f_2^T(\cdot), \cdots, f_n^T(\cdot)]^T \in \mathbb{R}^n$ in (1) is globally Lipschitz, i.e., $\|f_i(x,t) - f_i(y,t)\| \leq \gamma_i \|x(t) - y(t)\|$, for any $x, y \in \mathbb{R}^n$ and $\gamma_i > 0$ for $i = 1, 2, \cdots, N$.*

It should be mentioned that Assumption 2 holds when the Jacobian matrix $[\frac{\partial f}{\partial x}]_{n\times n}$ is uniformly bounded, which is the case for a number of well-known complex systems and complex networks, such as Chua's circuit [41], chaotic delayed system [42], neural networks [43], etc.

Lemma 1. *[44] For any vector $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$ and symmetric positive definite matrix W with proper dimensions, then $2x^T y \leq x^T W^{-1} x + y^T W y$.*

Lemma 2. *[45] For any constant matrix $0 \prec R \in \mathbb{R}^{n\times n}$, a scalar function $\tau(t)$ with $0 < \tau(t) \leq \bar{\tau}$ and vector function $\dot{x} : [-\bar{\tau}, 0] \to \mathbb{R}^n$ such that the integration concerned is well defined, let $\int_{t-\tau(t)}^{t} \dot{x}(s)ds = E\psi(t)$, where the matrix $E \in \mathbb{R}^{n\times k}$ and $\psi(t) \in \mathbb{R}^k$. Then the following inequality holds:*

$$-\int_{t-\tau(t)}^{t} \dot{x}^T(s) R \dot{x}(s) ds \leq \psi^T(t) \Upsilon_1 \psi(t),$$

where $\Upsilon_1 = -E^T G - G^T E + \tau(t) G^T R^{-1} G$ and $G \in \mathbb{R}^{n\times k}$.

C. Event-triggered control

In this subsection, the corresponding well-defined triggering condition is proposed and the error equations are derived by utilizing an event-triggered and pinning strategy.

Consider the sampling instants $0 = T_0 < T_1 < T_2 < \cdots < T_k < \cdots$ over the stochastic process $r(t)$. Obviously, the sampling period $T_{k+1} - T_k$ is stochastic. In this paper we assume that the sampling period $T_{k+1} - T_k$ takes its value in a finite set $\{h_1, h_2, \cdots, h_m\}$, where $0 = h_0 < h_1 < \cdots < h_m$, $m \geq 1$. At sampling instant T_k, for the i-th node in the heterogeneous network (1) the event detector is defined as

$$\delta_i^T(T_k)\delta_i(T_k) > \sigma_i z_i^T(T_k) z_i(T_k),\tag{5}$$

where the measurement error $\delta_i(T_k)$ is the difference between the state of the i-th node at the current sampled instant and the last event-triggered instant, i.e., $\delta_i(T_k) = x_i(T_k) - x_i(\hat{t}_k^i) \in \mathbb{R}^n$, with $\hat{t}_k^i = \max\{t_l^i : l \geq 0, t_l^i < T_k\}$ being the latest event-triggered instant before T_k for the i-th node. In addition, $\sigma_i > 0$ is a threshold parameter and

$$\begin{aligned}z_i(T_k) = &c\sum_{j=1}^{N} a_{ij}(r(T_k))\Gamma(x_j(T_k) - x_i(T_k))\\&+ ca_{i0}(r(T_k))\Gamma(s(T_k) - x_i(T_k)).\end{aligned}$$

If the condition (5) is satisfied at sampling instant T_k, an event is triggered for the i-th node, and the new event-triggered instant will be T_k. At the same time, $\delta_i(T_k)$ will be reset to zero. For the i-th node, the event-triggered intervals $t_1^i - t_0^i$, $t_2^i - t_1^i$, $\cdots$, $t_{k+1}^i - t_k^i$, $\cdots$ will not be shorter than $h_{\min} > 0$, where $h_{\min} = \min_{m\in\mathbb{N}^+}\{h_1, h_2, \cdots, h_m\}$. Hence, the positive lower-bound on the event-triggered intervals is obtained to guarantee that there is no Zeno behaviour existing in the proposed event-triggered scheme.

Remark 5. *Note that the event-triggered instants t_k^i $(k = 0, 1, 2, \cdots)$ belong to the set $\{0, T_1, \cdots, T_k, \cdots\}$, and the number of control actuation updates may be less than that with traditional sampled-data control. However, it should be pointed out that if the sampling period $T_{k+1} - T_k$ is relatively large or the parameter σ_i is too small, then the event might be triggered at each sampling instant. Besides, it is obvious that the event-triggered strategy becomes a time-triggered strategy when $\sigma_i = 0$.*

Considering the definitions of $x_i(\hat{t}_k^i)$ and $\delta_i(T_k)$, we know that if the event for the i-th node is not triggered at the instant T_k, then $t_k^i = \hat{t}_k^i$ and $x_i(t_k^i) = x_i(T_k) - \delta_i(T_k)$; otherwise, $t_k^i = T_k$, and $\delta_i(T_k)$ is reset to zero, i.e., $\delta_i(T_k) = 0$. Hence, $x_i(t_k^i) = x_i(T_k) - \delta_i(T_k)$ holds for all time. For $t \in [T_k, T_{k+1})$, $k = 0, 1, 2, \cdots$, the heterogeneous network (4) can then be changed into

$$\begin{aligned}\dot{x}_i(t) = &B_i x_i(t) + f(x_i(t),t) + c\sum_{j=1}^{N} a_{ij}(r(T_k))\\&\times \Gamma[x_j(T_k)-\delta_j(T_k)-(x_i(T_k)-\delta_i(T_k))]\\&+ ca_{i0}(r(T_k))\Gamma[s(T_k) - (x_i(T_k) - \delta_i(T_k))],\end{aligned}\tag{6}$$

where $i = 1, 2, \cdots, N$. Let $e_i(t) = x_i(t) - s(t)$, $i = 1, 2, \cdots, N$. Accordingly, when $r(T_k) = u \in \mathbb{S}$, we can obtain the following error system:

$$\begin{aligned}\dot{e}_i(t) &= B_i e_i(t) + f_e(e_i(t), t) \\ &\quad + c\sum_{j=1}^{N} a_{ij}(u)\Gamma[x_j(T_k) - \delta_j(T_k) - (x_i(T_k) - \delta_i(T_k))] \\ &\quad + c a_{i0}(u)\Gamma[s(T_k) - (x_i(T_k) - \delta_i(T_k))] + (B_i - B)s(t),\end{aligned} \tag{7}$$

where $f_e(e_i(t), t) = f(x_i(t), t) - f(s(t), t)$.

Additionally, we have that

$$\begin{aligned}&\sum_{j=1}^{N} a_{ij}(u)[x_j(T_k) - x_i(T_k)] \\ &= -\sum_{j=1, j\neq i}^{N} l_{ij}(u)x_j(T_k) - l_{ii}x_i(T_k) \\ &= -\sum_{j=1}^{N} l_{ij}(u)[x_j(T_k) - s(T_k) + s(T_k)] = -\sum_{j=1}^{N} l_{ij}(u)e_j(T_k),\end{aligned}$$

where $l_{ij}(u)$ denotes the elements of the Laplacian matrix $L(u)$ in the heterogeneous network, i.e., $L(u) = [l_{ij}(u)]_{N\times N}$, and $e_j(T_k)$ represents the synchronization error for node j at the instant T_k.

Define $W(s(t)) = [w_1^T(s(t)), w_2^T(s(t)), \cdots, w_N^T(s(t))]^T$, and $\sup_{t>T(s(0))} \|w_i(s(t))\| = \varphi_i$, $i = 1, 2, \cdots, N$, where $w_i(s(t)) = (B_i - B)s(t)$. Then let $e(t) = [e_1^T(t), e_2^T(t), \cdots, e_N^T(t)]^T$ and

$$\begin{aligned}e(T_k) &= [e_1^T(T_k), e_2^T(T_k), \cdots, e_N^T(T_k)]^T, \\ \delta(T_k) &= [\delta_1^T(T_k), \delta_2^T(T_k), \cdots, \delta_N^T(T_k)]^T, \\ F_e(e(t), t) &= [f_e^T(e_1(t), t), \cdots, f_e^T(e_N(t), t)]^T.\end{aligned}$$

For $t \in [T_k, T_{k+1}), k = 0, 1, 2, \cdots$ and $u \in \mathbb{S}$, by the property of Kronecker product, the error system (7) can be rewritten in compact form as

$$\begin{aligned}\dot{e}(t) &= \mathbb{B}e(t) + F_e(e(t), t) - c(H(u)\otimes\Gamma)e(T_k) \\ &\quad + c(H(u)\otimes\Gamma)\delta(T_k) + W(s(t)),\end{aligned} \tag{8}$$

where $\mathbb{B} = \text{diag}\{B_1, B_2, \cdots, B_N\}$, $H(u) = L(u) + A_0(u)$.

Based on the above discussions, we introduce $\tau(t) = t - T_k$ to the error system (8), then we have

$$\begin{aligned}\dot{e}(t) &= \mathbb{B}e(t) + F_e(e(t), t) - c(H(u)\otimes\Gamma)e(t - \tau(t)) \\ &\quad + c(H(u)\otimes\Gamma)\delta(t - \tau(t)) + W(s(t)),\end{aligned} \tag{9}$$

where $t \in [T_k, T_{k+1}), k = 0, 1, 2, \cdots$.

Consider $\tau(t) = t - T_k \in [0, T_{k+1} - T_k)$ and the sampling period $T_{k+1} - T_k$ takes values in a finite set $\{h_1, h_2, \cdots, h_m\}$. We have that $\tau(t)$ can be divided into m segments, i.e., $\tau_1(t) \in [0, h_1)$, $\tau_2(t) \in [h_1, h_2)$, $\cdots$, $\tau_m(t) \in [h_{m-1}, h_m)$. Hence the following stochastic variables are defined for $l = 1, 2, \cdots, m$,

$$\beta_l(t) = \begin{cases} 1, & h_{l-1} \le \tau_l(t) < h_l \\ 0, & \text{otherwise} \end{cases}$$

and $\mathbb{P}\{\beta_l(t) = 1\} = \mathbb{P}\{h_{l-1} \le \tau_l(t) < h_l\} = \beta_l$, where $\beta_l \in (0, 1)$ and $\sum_{l=1}^{m} \beta_l = 1$. Moveover, $\mathbb{E}[\beta_l(t)] = \beta_l$, $\mathbb{E}[\beta_l(t) - \beta_l]^2 = \beta_l(1 - \beta_l)$. Thus the error system (9) can be changed into

$$\begin{aligned}\dot{e}(t) &= \mathbb{B}e(t) + F_e(e(t), t) - \sum_{l=1}^{m} \beta_l(t)(c(H(u)\otimes\Gamma))e(t - \tau_l(t)) \\ &\quad + \sum_{l=1}^{m} \beta_l(t)(c(H(u)\otimes\Gamma))\delta(t - \tau_l(t)) + W(s(t)),\end{aligned} \tag{10}$$

where $t \in [T_k, T_{k+1})$, and $h_{l-1} \le \tau_l(t) < h_l$, $l = 1, 2, \cdots, m$. Meanwhile, the initial condition of $e(t)$ is supplied as $e(t) = \psi(t)$ for $t \in [-h_m, 0]$. $\psi(t) \in L^2_{\mathscr{F}_0} C([-h_m, 0], \mathbb{R}^{Nn})$, which is to denote the family of all $\mathscr{F}_0$-measurable $C([-h_m, 0], \mathbb{R}^{Nn})$-valued random variables satisfying $\sup_{-h_m \le \mu \le 0} \mathbb{E}(\|f(\mu)\|^2) < \infty$.

III. Main Results

A. Pinning strategy

In this subsection, a search algorithm with a linear time complexity is proposed for choosing the nodes to be pinned in heterogeneous network (4). Some related approaches on pinning synchronization with deterministic switching have been provided for homogeneous networks in [46], [47]. However, when it comes to heterogeneous networks with stochastic switching, methods on pinning synchronization have not been properly investigated yet. In this paper, we propose a pinning strategy algorithm to determine how many and which nodes should be pinned for random switching heterogeneous networks such that Assumption 1 holds.

It is assumed that the stochastic switching among the different topologies is triggered by the sudden loss or recovery of communication links, and there is no node that will be deleted from the heterogeneous network. For $r(t) = u \in \mathbb{S}$ and $t \in [T_k, T_{k+1})$, let $\mathcal{G}(u) = (\mathcal{V}(u), E(u), A(u))$ be the communication topology in mode u. The set $\mathcal{V}_{\text{pin}}(u) = \{i_1(u), \cdots, i_l(u)\}$ of pinned nodes is searched on a generalized Markovian switching topology of heterogeneous network by the following procedures. Namely, Assumption 1 will be satisfied if these $l(u)$ nodes searched by the following **Algorithm 1** are selected and pinned in the heterogeneous network.

Algorithm 1 Pinning nodes search algorithm

1) For $\mathcal{G}(u)$, find out all the nodes with zero in-degree and strongly connected components. Assume that there are $o_1(u)(o_1(u) \ge 0)$ nodes with zero in-degree, labeled as $i_1(u), i_2(u), \cdots, i_{o_1}(u)$, and $o_2(u)(o_2(u) \ge 0)$ strongly connected components, represented by $\mathcal{G}(\mathcal{V}_1(u), E_1(u), A_1(u))$, $\cdots$, $\mathcal{G}(\mathcal{V}_{o_2}(u), E_{o_2}(u), A_{o_2}(u))$ in $\mathcal{G}(u)$. Set $l(u) = 0$ and $g(u) = 1$.

2) All the $o_1(u)$ nodes with zero in-degree should be selected and pinned. Then update the value $l(u)$ by $l(u) = l(u) + o_1(u)$. If $o_2(u) = 0$, stop; else go to step **3)**.

3) Check whether there exists at least one node in $\mathcal{V}_g(u)$ which is reachable from a node belonging to the node set $\mathcal{V} \setminus \mathcal{V}_g(u)$. If there exist such nodes, go to step **4)**; otherwise, go to **5)**.

4) If $g(u) < o_2(u)$, let $g(u) = g(u) + 1$ and re-perform step **3)**; otherwise stop.

5) Arbitrarily choose one node in $\mathcal{V}_g(u)$ to be pinned, update the value of $l(u)$ by $l(u) = l(u) + 1$. Then re-perform step **4)**.

The objective of **Algorithm 1** is to choose appropriate pinning nodes to guarantee that the augmented graph $\tilde{\mathcal{G}}(u) = (\tilde{\mathcal{V}}, \tilde{E}(u), \tilde{A}(u))$ contains a directed spanning tree with the virtual node $N+1$ as the root. In other words, if there are fewer than $l(u)$ nodes in $\mathcal{G}(u) = (\mathcal{V}(u), E(u), A(u))$ that are selected and pinned, then it can be checked that there exists at least one node in $\mathcal{G}(u)$ which is not reachable by the virtual node $N+1$.

Remark 6. *The proposed **Algorithm 1** belongs to the classical pinning control approach requiring centralized knowledge of the network topology to detect whether there are possible coupling and control gains ensuring pinning controllability [48]. Actually, the virtual leader node $s(t)$ may be regarded as playing the role of the central controller that is capable of monitoring the change of network topology and deciding on the selection of the corresponding pinning nodes in the underlying topology structure. The nodes in heterogeneous networks may not need to have such kind of global information, though they may need to report the local information to the virtual leader node for composing the global information and making decisions. Distributed pinning schemes and their designs are out of the scope of this paper, and will be investigated in our future studies.*

Next, we present a simple example to demonstrate how to find the pinning nodes in a given directed graph with Markovian switching such that the virtual node $N+1$ has a path to every other nodes. Given a finite state space $\mathbb{S} = \{1, 2, 3\}$, as illustrated in Fig. 1, the network $\mathcal{G}(\mathcal{V}, E(1), A(1))$ contains 11 nodes and node 12 is a virtual leader. According to step **1** in **Algorithm 1**, it can be seen that there are two nodes 7 and 11 with zero in-degree, and two strongly connected components. The algorithm thus goes to step **2**, where node 7 and node 11 are pinned. By the remainder steps, it can be obtained that two distinct nodes arbitrarily selected from node sets $\{1, 2, 3, 4\}$ and $\{8, 9, 10\}$ respectively should be pinned. Assume we select node 4 and node 8. The pinning set $\mathcal{V}_{\text{pin}}(1) = \{4, 7, 8, 11\}$ guarantees that the virtual node 12 has a path to every node. Similarly, $\mathcal{V}_{\text{pin}}(2) = \{3, 7, 9\}$ and $\mathcal{V}_{\text{pin}}(3) = \{3, 6, 9, 10, 11\}$ as shown in Fig. 2 and Fig. 3, respectively.

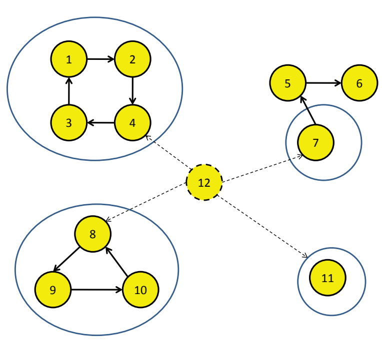

Fig. 1. Pinning strategy of a heterogeneous Markovian switching network in mode "1".

Therefore, given the network topology graph in mode $u \in \mathbb{S}$, the pinning set $\mathcal{V}_{\text{pin}}(u) = \{i_1(u), \cdots, i_l(u)\}$ can be determined by the proposed algorithm. Next, we consider the strategy for imposing pinning feedback gains over the pinning nodes $i_1(u), \cdots, i_l(u)$.

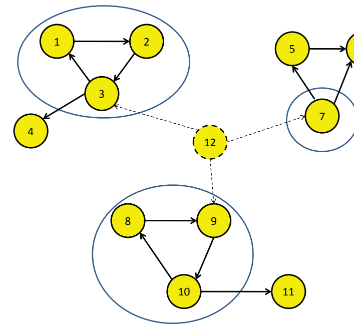

Fig. 2. Pinning strategy of a heterogeneous Markovian switching network in mode "2".

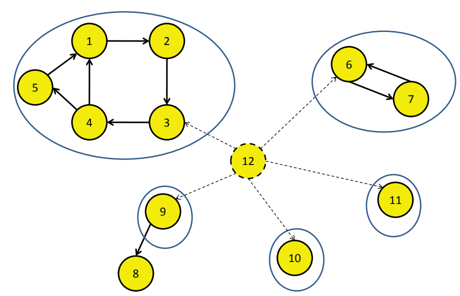

Fig. 3. Pinning strategy of a heterogeneous Markovian switching network in mode "3".

B. The strategy of pinning feedback gains

In this subsection, an efficiency restriction method is employed to obtain the proper pinning feedback gains $a_{i_1 0}(u)$, $a_{i_2 0}(u)$, $\cdots$, $a_{i_l 0}(u)$ for the selected pinning nodes $i_1(u)$, $i_2(u)$, $\cdots$, $i_l(u)$, where $u \in \mathbb{S} = \{1, 2, \cdots, M\}$. A condition is proposed for properly selecting the coupling strength and the inner coupling matrix.

Considering the constraint of control efficiency widely existing in many applications, we propose a hypothesis that the heterogeneous network has limited control efficiency [49], which follows

$$\sum_{i_k(u) \in \mathcal{V}_{\text{pin}}(u)} a_{i_k 0}(u) = E_0, \quad u \in \mathbb{S} = \{1, 2, \cdots, M\}. \quad (11)$$

Motivated by the result of [49], to find the pinning feedback gains with a fixed control efficiency E_0, we can obtain the feasible solution that

$$a_{i_1 0}(u) = a_{i_2 0}(u) = \cdots = a_{i_l 0}(u) = \frac{E_0}{l(u)} \triangleq d(u).$$

Remark 7. *We assume that a switching heterogeneous network may have fixed control efficiency E_0 in practice. Subject to such a constraint, we consider the case where all selected pinning nodes have an identical feedback gain $\frac{E_0}{l(u)}$. In [49], It has been proven that the average distributed solution can be feasible and it may be the optimal pinning feedback gain under certain conditions.*

Without loss of generality, we rearrange the order of nodes in the network such that the pinned nodes $i_1(u), \cdots, i_l(u)$ are the first $l(u)$ nodes in the rearranged network. Then we have

$$A_0(u) = \mathrm{diag}\{\underbrace{d(u), \cdots, d(u)}_{l(u)}, \underbrace{0, \cdots, 0}_{N-l(u)}\}.$$

It is well known that the synchronization behavior of network may be sensitive to the inner coupling matrix and the coupling strength, which should not be too large or too small for achieving synchronization in the network. One has to carefully design the coupling strength and the inner coupling matrix. Next we present a criterion on selecting the coupling strength c and inner coupling matrix Γ under limited control efficiency.

For simplicity, we assume that $B_i = B$ for $i = 1, 2, \cdots, N$ in the equation (1). Then we can obtain that

$$\dot{e}_i(t) = Be_i(t) + f_e(e_i(t), t) - c\sum_{j=1}^{N} l_{ij}(u)\Gamma e_j(t) - ca_{i0}(u)\Gamma e_i(t), \tag{12}$$

where $u \in \mathbb{S}$, $f_e(e_i(t), t) = f(x_i(t), t) - f(s(t), t)$.

Corollary 1. *Suppose that $\bar{\gamma} = \max\{\gamma_1, \cdots, \gamma_N\}$, where $\gamma_1, \cdots, \gamma_N$ are defined in Assumption 2, the error network (12) under limited control efficiency can be stable if the coupling strength c and inner coupling matrix Γ are selected to satisfy the following inequalities*

$$\left(\lambda_{\max}\{\frac{1}{2}\mathrm{He}(B)\} + \bar{\gamma}\right) I_N - c\alpha_j \left[\frac{1}{2}\mathrm{He}(L(u)) + A_0(u)\right] \prec 0, \tag{13}$$

for all $j = 1, 2, \cdots, n$, $u \in \mathbb{S}$.

Proof. The proof is reported in Appendix A for the sake of legibility. □

Remark 8. *It should be mentioned that Corollary 1 is a sufficient condition under which the special case of heterogeneous network can be stable. In a sense, it has provided a way to properly choose the coupling strength c and the inner coupling matrix Γ for given pinning feedback gains.*

C. Quasi-synchronization analysis

In this subsection, the sufficient conditions are established to guarantee quasi-synchronization behavior for the heterogeneous network with a generalized Markovian topology.

Theorem 1. *Suppose the generalized Markovian switching heterogeneous network has limited control efficiency E_0 governed by (13). For given scalars $\sigma_i > 0$, $\varphi_i > 0$, $h_l > 0$, $\beta_l \in (0, 1)$ and $\sum_{l=1}^{m} \beta_l = 1$, where $i = 1, 2, \cdots, N$, $l = 1, 2, \cdots, m$, under Assumption 1 and Assumption 2, the trajectory of the error system (10) converges exponentially into a ball $\mathcal{M}$ at a convergence rate θ, where $\mathcal{M} = \{e(t) \in \mathbb{R}^{Nn} : \|e(t)\| \leq \varpi\}$ as $t \to \infty$, $\varpi = \sqrt{\frac{\bar{\kappa}\sum_{i=1}^{N}\varphi_i^2}{2\theta\underline{\kappa}}}$, if there exist positive definite matrices $P(u) \succ 0$, $R(u) \succ 0$, $u \in \mathbb{S}$, $Q_l \succ 0$, $X_l \succ 0$, $Y_l \succ 0$, and matrices J_{1l}, J_{2l} with appropriate dimensions such that*

$$\Omega(u) + \sum_{l=1}^{m} \beta_l \vartheta_l J_{1l}^T X_l^{-1} J_{1l} + \sum_{l=1}^{m} \beta_l \vartheta_l J_{2l}^T Y_l^{-1} J_{2l} \prec 0, \tag{14}$$

for all $u \in \mathbb{S}$, $\vartheta_l = e^{-2\theta h_l}(h_l - h_{l-1})$ and

$$\begin{aligned}\Omega(u) &= Z_1^T \big[\sum_{v=1}^{M} \pi_{uv} P(v) + \bar{\gamma} I + 2\theta P(u) + \beta_1 Q_1 + P(u) \\ &\times R^{-1}(u) P(u)\big] Z_1 + \mathrm{He}\big\{Z_1^T P(u) \Pi(u) - \sum_{l=1}^{m} \beta_l e^{-2\theta h_l} \\ &\times (Z_{l+1} - Z_{m+l+1})^T J_{1l} - \sum_{l=2}^{m} \beta_l e^{-2\theta h_l} (Z_{m+l} - Z_{l+1})^T \\ &\times J_{2l} - \beta_1 (Z_1 - Z_2)^T J_{21}\big\} - \beta_1 e^{-2\theta h_1} Z_{m+2}^T Q_1 Z_{m+2} \\ &+ \sum_{l=2}^{m} \beta_l e^{-2\theta h_{l-1}} Z_{m+l}^T Q_l Z_{m+l} - \sum_{l=1}^{m} \beta_l Z_{2m+l+1}^T Z_{2m+l+1} \\ &- \sum_{l=2}^{m} \beta_l e^{-2\theta h_{l-1}} Z_{m+l+1}^T Q_l Z_{m+l+1} - Z_{3m+2}^T Z_{3m+2} \\ &+ \sum_{l=1}^{m} \beta_l Z_{l+1}^T ((H^T(u) \Lambda H(u)) \otimes \Gamma) Z_{l+1} \\ &+ \sum_{l=1}^{m} \beta_l (h_l - h_{l-1}) \Pi^T(u) (X_l + Y_l) \Pi(u), \quad \Pi(u) = \mathbb{B}(u) \\ &\times Z_1 + Z_{3m+2} + \sum_{l=1}^{m} \beta_l (cH(u) \otimes \Gamma)(Z_{2m+l+1} - Z_{l+1}),\end{aligned}$$

where $Z_k = [0, \cdots, 0, I_{Nn}, 0, \cdots, 0] \in \mathbb{R}^{Nn \times (3m+2)Nn}$ is a block matrix with $3m+2$ block elements, in which the kth block element is I_{Nn} and the other block elements are zero matrices. Moreover, $\bar{\kappa} = \lambda_{\max, u\in\mathbb{S}}\{R(u)\}$, $\underline{\kappa}_1 = \lambda_{\min, u\in\mathbb{S}}\{P(u)\}$, $\underline{\kappa}_2 = \lambda_{\min}\{Q_l\}$, $\underline{\kappa}_3 = \lambda_{\min}\{X_l\}$, $\underline{\kappa}_4 = \lambda_{\min}\{Y_l\}$ and

$$\begin{aligned}\Lambda &= \mathrm{diag}\{\sigma_1, \sigma_2, \cdots, \sigma_N\} \in \mathbb{R}^{N\times N}, \\ \underline{\kappa} &= \underline{\kappa}_1 + \sum_{l=1}^{m} \beta_l \vartheta_l [\underline{\kappa}_2 + \frac{1}{2}(h_l + h_{l-1})(\underline{\kappa}_3 + \underline{\kappa}_4)].\end{aligned}$$

Proof. Consider the following Lyapunov functional candidate

$$V(e(t), t, u) = V_1(e(t), t, u) + V_2(e(t), t, u) + V_3(e(t), t, u), \tag{15}$$

where $u \in \mathbb{S}$, $t \in [T_k, T_{k+1})$, $V_1(e(t), t, u) = e^T(t) P(u) e(t)$, $V_2(e(t), t, u) = \sum_{l=1}^{m} \beta_l \int_{t-h_l}^{t-h_{l-1}} e^{2\theta(s-t)} e^T(s) Q_l e(s) ds$ and $V_3(e(t), t, u) = \sum_{l=1}^{m} \beta_l \int_{-h_l}^{-h_{l-1}} \int_{t+\mu}^{t} e^{2\theta(s-t)} \dot{e}^T(s)(X_l + Y_l)\dot{e}(s) ds d\mu$.

Let $\mathscr{L}$ be the weak infinitesimal operator of the Lyapunov functional $V(e(t), t, u)$. Based on the definition of infinitesimal operator [50], along the trajectory of (10) we obtain that $\mathscr{L}V(e(t),t,u)=\sum_{k=1}^{3}\mathscr{L}V_k(e(t),t,u)$ and

$$\begin{aligned}\mathscr{L}V_1(e(t),t,u)=&\sum_{v=1}^{M}\pi_{uv}e^T(t)P(v)e(t)-2\theta V_1(e(t),t,u)\\&+2\theta e^T(t)P(u)e(t)+2e^T(t)P(u)\Big\{\mathbb{B}(u)e(t)\\&+F_e(e(t),t)-\sum_{l=1}^{m}\beta_l(t)(cH(u)\otimes\Gamma)e(t-\tau_l(t))\\&+\sum_{l=1}^{m}\beta_l(t)(cH(u)\otimes\Gamma)\delta(t-\tau_l(t))+W(u,s(t))\Big\},\end{aligned}\tag{16}$$

$$\begin{aligned}\mathscr{L}V_2(e(t),t,u)=&\sum_{l=1}^{m}\beta_l\big[e^{-2\theta h_{l-1}}e^T(t-h_{l-1})Q_le(t-h_{l-1})\\&-e^{-2\theta h_l}e^T(t-h_l)Q_le(t-h_l)\big]-2\theta V_2(e(t),t,u),\end{aligned}\tag{17}$$

$$\begin{aligned}\mathscr{L}V_3(e(t),t,u)=&\sum_{l=1}^{m}\beta_l(h_l-h_{l-1})\dot{e}^T(t)(X_l+Y_l)\dot{e}(t)\\&-\sum_{l=1}^{m}\beta_l\int_{t-h_l}^{t-h_{l-1}}e^{2\theta(s-t)}\dot{e}^T(s)(X_l+Y_l)\dot{e}(s)ds\\&-2\theta V_3(e(t),t,u).\end{aligned}\tag{18}$$

Define

$$\begin{aligned}\xi(t)=[&e^T(t),e^T(t-\tau_1(t)),\cdots,e^T(t-\tau_m(t)),\\&e^T(t-h_1),\cdots,e^T(t-h_m),\delta^T(t-\tau_1(t)),\cdots,\\&\delta^T(t-\tau_m(t)),F_e^T(e(t),t)]^T.\end{aligned}$$

According to Lemma 2, we can obtain that

$$\begin{aligned}&-\sum_{l=1}^{m}\beta_l\int_{t-h_l}^{t-h_{l-1}}e^{2\theta(s-t)}\dot{e}^T(s)(X_l+Y_l)\dot{e}(s)ds\\\le&\ \xi^T(t)\Big\{\sum_{l=1}^{m}\beta_l\vartheta_lJ_{1l}^TX_l^{-1}J_{1l}+\sum_{l=1}^{m}\beta_l\vartheta_lJ_{2l}^TY_l^{-1}J_{2l}\\&-\sum_{l=1}^{m}\beta_le^{-2\theta h_l}\,\mathrm{He}\{(Z_{l+1}-Z_{m+l+1})^TJ_{1l}\}\\&-\sum_{l=2}^{m}\beta_le^{-2\theta h_l}\,\mathrm{He}\{(Z_{m+l}-Z_{l+1})^TJ_{2l}\}\\&-\mathrm{He}\{\beta_1J_{21}^T(Z_1-Z_2)\}\Big\}\xi(t).\end{aligned}\tag{19}$$

Therefore, combining (16)-(19), we have

$$\begin{aligned}&\mathscr{L}V(e(t),t,u)\le\mathscr{L}V_1(e(t),t,u)+\mathscr{L}V_2(e(t),t,u)\\&+\xi^T(t)\Big\{\sum_{l=1}^{m}\beta_l\vartheta_lJ_{1l}^TX_l^{-1}J_{1l}+\sum_{l=1}^{m}\beta_l\vartheta_lJ_{2l}^TY_l^{-1}J_{2l}\\&-\sum_{l=1}^{m}\beta_le^{-2\theta h_l}\,\mathrm{He}\{(Z_{l+1}-Z_{m+l+1})^TJ_{1l}\}\\&-\sum_{l=2}^{m}\beta_le^{-2\theta h_l}\,\mathrm{He}\{(Z_{m+l}-Z_{l+1})^TJ_{2l}\}\\&-\mathrm{He}\{\beta_1J_{21}^T(Z_1-Z_2)\}\Big\}\xi(t)-2\theta V_3(e(t),t,u)\\&+\sum_{l=1}^{m}\beta_l(h_l-h_{l-1})\dot{e}^T(t)(X_l+Y_l)\dot{e}(t)\\&=-2\theta V(e(t),t,u)+\sum_{l=1}^{m}\beta_l(h_l-h_{l-1})\dot{e}^T(t)(X_l+Y_l)\dot{e}(t)\\&+\xi^T(t)\Theta(u)\xi(t)+2e^T(t)P(u)W(u,s(t)),\end{aligned}\tag{20}$$

where

$$\begin{aligned}\Theta(u)=&\sum_{l=1}^{m}\beta_l\vartheta_lJ_{1l}^TX_l^{-1}J_{1l}+\sum_{l=1}^{m}\beta_l\vartheta_lJ_{2l}^TY_l^{-1}J_{2l}\\&+Z_1^T\Big[\sum_{v=1}^{M}\pi_{uv}P(v)+2\theta P(u)\Big]Z_1+\beta_1Z_1^TQ_1Z_1\\&-\beta_1e^{-2\theta h_1}Z_{m+2}^TQ_1Z_{m+2}+\sum_{l=2}^{m}\beta_le^{-2\theta h_{l-1}}Z_{m+l}^TQ_lZ_{m+l}\\&-\sum_{l=2}^{m}\beta_le^{-2\theta h_{l-1}}Z_{m+l+1}^TQ_lZ_{m+l+1}+\mathrm{He}\Big\{Z_1^TP(u)Z_{3m+2}\\&+\sum_{l=1}^{m}\beta_l(t)Z_1^TP(u)(cH(u)\otimes\Gamma)(Z_{2m+l+1}-Z_{l+1})\\&-\sum_{l=1}^{m}\beta_le^{-2\theta h_l}(Z_{l+1}-Z_{m+l+1})^TJ_{1l}-\beta_1(Z_1-Z_2)^TJ_{21}\\&-\sum_{l=2}^{m}\beta_le^{-2\theta h_l}(Z_{m+l}-Z_{l+1})^TJ_{2l}\Big\}.\end{aligned}$$

According to Lemma 1, it follows that

$$\begin{aligned}&2e^T(t)P(u)W(u,s(t))\\\le&\ e^T(t)P(u)R^{-1}(u)P(u)e(t)+W^T(u,s(t))R(u)W(u,s(t))\\\le&\ e^T(t)P(u)R^{-1}(u)P(u)e(t)+\lambda_{\max,u\in\mathbb{S}}\{R(u)\}\sum_{i=1}^{N}\varphi_i^2.\end{aligned}\tag{21}$$

Considering the event-triggering condition (5), it can be obtained that

$$\begin{aligned}\xi^T(t)\Big[&\sum_{l=1}^{m}\beta_lZ_{l+1}^T((H^T(u)\Lambda H(u))\otimes\Gamma)Z_{l+1}\\&-\sum_{l=1}^{m}\beta_lZ_{2m+l+1}^TZ_{2m+l+1}\Big]\xi(t)\ge0.\end{aligned}\tag{22}$$

Considering the Assumption 2 with respect to vector-valued function $f(\cdot)$, we have

$$\xi^T(t)(\bar{\gamma}Z_1^TZ_1-Z_{3m+2}^TZ_{3m+2})\xi(t)\ge0.\tag{23}$$

Based on the equation (20), with the aid of (10) and (21)-(23), we can obtain that

$$\begin{aligned}\mathbb{E}\{\mathscr{L}V(e(t),t,u)\}\le&-2\theta\mathbb{E}\{V(e(t),t,u)\}+\sum_{l=1}^{m}\beta_l(h_l-h_{l-1})\\&\times\mathbb{E}\{\dot{e}^T(t)(X_l+Y_l)\dot{e}(t)\}+\mathbb{E}\{\xi^T(t)\Theta(u)\xi(t)\}\\&+\mathbb{E}\{2e^T(t)P(u)W(u,s(t))\}\end{aligned}$$

$$\leq -2\theta\mathbb{E}\{V(e(t),t,u)\} + \overline{\kappa}\sum_{i=1}^{N}\varphi_i^2 + \mathbb{E}\{\xi^T(t)[\Omega(u) + \sum_{l=1}^{m}\beta_l\vartheta_l J_{1l}^T X_l^{-1} J_{1l} + \sum_{l=1}^{m}\beta_l\vartheta_l J_{2l}^T Y_l^{-1} J_{2l}]\xi(t)\}. \quad (24)$$

By (14), it follows that

$$\mathbb{E}\{\mathscr{L}V(e(t),t,u)\} \leq -2\theta\mathbb{E}\{V(e(t),t,u)\} + \overline{\kappa}\sum_{i=1}^{N}\varphi_i^2. \quad (25)$$

Multiplying (25) by $e^{2\theta t}$ and integrating it from 0 to t, it yields

$$\mathbb{E}\{V(e(t),t,u)\} \leq V(e(0),r(0))e^{-2\theta t} + \frac{1-e^{-2\theta t}}{2\theta}\overline{\kappa}\sum_{i=1}^{N}\varphi_i^2.$$

Noticing $\|e(t)\|^2\underline{\kappa} \leq \mathbb{E}\{V(e(t),t,u)\}$, we have

$$\|e(t)\| \leq \sqrt{\frac{V(e(0),r(0))}{\underline{\kappa}}}e^{-\theta t} + \sqrt{1-e^{-2\theta t}}\sqrt{\frac{\overline{\kappa}\sum_{i=1}^{N}\varphi_i^2}{2\theta\underline{\kappa}}}.$$

If sampling instant $T_k \to \infty$, then $t \to \infty$ and the error system (10) converges exponentially into a ball $\mathcal{M}$ at a convergence rate θ. This completes the proof. □

Remark 9. *It should be mentioned that multiple Lyapunov functionals* (15) *are explicitly constructed to deal with the quasi-synchronization of heterogeneous networks with generalized Markovian switching topologies and event-triggered communication, which leads to a less conservative result than the single Lyapunov functional method. Moreover, the designed multiple Lyapunov functionals can be more complicated than* (15)*. However, it will be more challenging to calculate the weak infinitesimal operator when more complicated Lyapunov functional is applied.*

Remark 10. *It is challenging to achieve exponential quasi-synchronization in a heterogeneous network with generalized Markovian topology. Compared with the results presented in references [18], [19], [21], [25], Theorem 1 presents an explicit expression of the error bound. However, it should be pointed out that the quasi-synchronization criterion provided in our paper are dependent on the solvability of some high dimensional matrix inequalities, which may be inapplicable to switching heterogeneous networks of huge size.*

Due to the existence of the nonlinear terms $J_{1l}^T X_l^{-1} J_{1l}$ and $J_{2l}^T Y_l^{-1} J_{2l}$, the matrix inequality (14) may be difficult to be solved. To compute more conveniently, the following Corollary 2 is addressed instead of Theorem 1, where the sufficient condition is presented in terms of LMIs that can be easily solved numerically.

Corollary 2. *Suppose the generalized Markovian switching heterogeneous network has limited control efficiency E_0 governed by (13). For given scalars $\sigma_i > 0$, $\varphi_i > 0$, $h_l > 0$, $\beta_l \in (0,1)$ and $\sum_{l=1}^{m}\beta_l = 1$, where $i = 1,2,\cdots,N$, $l = 1,2,\cdots,m$, under Assumption 1 and Assumption 2, the trajectory of the error system (10) converges exponentially into a ball $\mathcal{M}$ at a convergence rate θ and $\mathcal{M} = \{e(t) \in \mathbb{R}^{Nn} : \|e(t)\| \leq \varpi\}$ as $t \to \infty$, where $\varpi = \sqrt{\frac{\overline{\kappa}\sum_{i=1}^{N}\varphi_i^2}{2\theta\underline{\kappa}}}$, if there exist positive definite matrices $P(u) \succ 0$, $R(u) \succ 0$, $u \in \mathbb{S}$, $Q_l \succ 0$, $X_l \succ 0$, $Y_l \succ 0$, and matrices J_{1l}, J_{2l} with appropriate dimensions such that*

$$\begin{bmatrix} \Omega(u) & \star & \star \\ \vartheta_l J_{1l} & -\vartheta_l X_l & \star \\ \vartheta_l J_{2l} & 0 & -\vartheta_l Y_l \end{bmatrix} \prec 0. \quad (26)$$

Proof. Since $\beta_l \in (0,1), l = 1,2,\cdots,m$ and $\sum_{l=1}^{m}\beta_l = 1$, we can rewrite (14) into the following form:

$$\sum_{l=1}^{m}\beta_l[\Omega(u) + \vartheta_l J_{1l}^T X_l^{-1} J_{1l} + \vartheta_l J_{2l}^T Y_l^{-1} J_{2l}] \prec 0.$$

For all $l = 1,2,\cdots,m$, if we let

$$\Omega(u) + \vartheta_l J_{1l}^T X_l^{-1} J_{1l} + \vartheta_l J_{2l}^T Y_l^{-1} J_{2l} \prec 0, \quad (27)$$

it can be seen that (14) holds. By the Schur complement lemma, (27) is equivalent to (26). This completes the proof. □

IV. Numerical Examples

Numerical examples are provided in this section to demonstrate the effectiveness of the proposed design scheme.

Example 1. *Suppose that a heterogeneous network consists of three dynamical nodes described by (1), where $x_i(t) = [x_{i1}(t), x_{i2}(t), x_{i3}(t)]^T$, $f(x_i(t)) = [|x_{i1}+1|-|x_{i1}-1|, 0, 0]^T$, $i = 1,2,3$ and*

$$B_1 = \begin{bmatrix} -3 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 0 & 0.1 \end{bmatrix}, \ B_2 = \begin{bmatrix} -3 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 0 & -0.1 \end{bmatrix},$$

$$B_3 = \begin{bmatrix} -3 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 0.1 & 0 \end{bmatrix}, \ B = \begin{bmatrix} -3 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The dynamics of virtual leader $s(t) = [s_{01}(t), s_{02}(t), s_{03}(t)]^T$ satisfies the equation (3).

In this numerical example, the graph topologies are governed by a generalized Markov process $r(t) \in \mathbb{S} = \{1,2\}$, where the transition rate $\pi_{11} = -1$, $\pi_{12} = 1$, $\pi_{21} = 1$, $\pi_{22} = -1$ for $t \in [T_k, T_{k+1})$. The switching topology of the network is shown in Fig. 4, where the Laplacian matrix can be given as follows

$$L(1) = \begin{bmatrix} 2 & 0 & -2 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \quad L(2) = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 2 \end{bmatrix}.$$

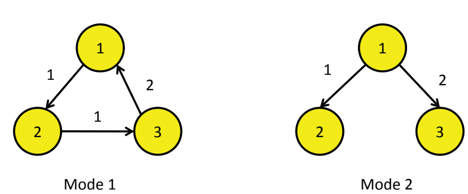

Fig. 4. Switching topology of the network in Example 1.

According to the proposed pinning strategy, we know that any of the three nodes can be selected as the pinning node in mode 1 and node 1 is the pinning node in mode 2. Without loss of generality, we choose node 2 as pinning node in mode 1. Assume that the heterogeneous network has limited control efficiency $E_0 = 5$. Then we have $A_0(1) = \text{diag}\{0, 5, 0\}$ and $A_0(2) = \text{diag}\{5, 0, 0\}$. Based on the condition (13) in Corollary 1, we choose the coupling strength as $c = 2$ and inner coupling matrix $\Gamma = \text{diag}\{6, 6, 6\}$. In this way, we obtain all the parameters in (9).

In simulation, the trajectory of virtual leader node is given by (3) with $s(0) = [-45.7167, 1.4624, -30.0280]^T$ and the initial states of nodes are selected as follows:

$$x_1(0) = [35.8114, 7.8239, -41.5736]^T,$$
$$x_2(0) = [-29.2207, -45.9492, -15.5741]^T,$$
$$x_3(0) = [46.4288, -34.9129, -43.3993]^T.$$

Adopting the stochastic sampling on interval $[0.009, 0.01]$ with a uniform distribution, the state trajectories of the virtual leader node and network nodes can be obtained as shown in Figs. 5-7, which indicate that the quasi-synchronization in generalized Markovian switching network is achieved by using the proposed design method. In Fig. 8, the events of each node under the proposed event-triggered approach are marked within the time interval $[0, 1.6]$, from which we can see that the sampling is sporadic rather than at every time instant, especially in later part of the time.

Define $\|e(t)\| = \sqrt{\sum_{i=1}^{3} \|x_i(t) - s(t)\|^2}$ as the quasi-synchronization error. We depict the corresponding simulated errors at different time t (i.e., the evolution of $\|e(t)\|$) as shown in Fig. 9. For the parameters adopting in Example 1, according to Corollary 2, one can obtain that the theoretical error bound $\varpi = 3.8987$. From Fig. 9, it can be seen that the nodes in switching network and the virtual leader are quasi-synchronized within the prescribed error bound $\varpi = 3.8987$ when $t > 0.3$, which indicates that a good performance and a fast convergence rate have been yielded by utilizing the proposed design method.

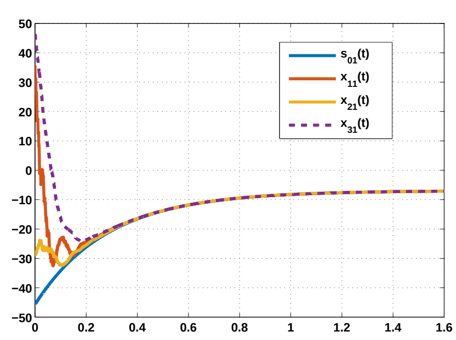

Fig. 5. State trajectories of $s_{01}(t)$, $x_{11}(t)$, $x_{21}(t)$, $x_{31}(t)$ in Example 1.

To compare the proposed control method versus the distributed impulsive control, we simulate distributed impulsive control by setting the impulsive interval be $[0.009, 0.01]$ with

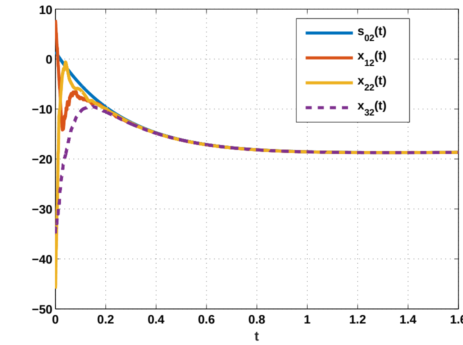

Fig. 6. State trajectories of $s_{02}(t)$, $x_{12}(t)$, $x_{22}(t)$, $x_{32}(t)$ in Example 1.

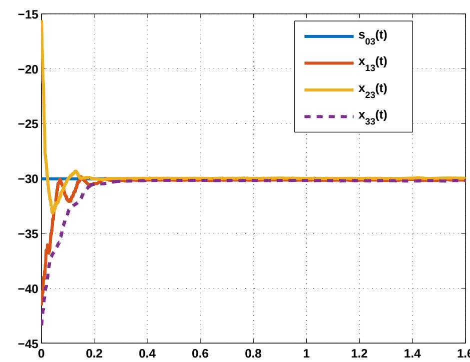

Fig. 7. State trajectories of $s_{03}(t)$, $x_{13}(t)$, $x_{23}(t)$, $x_{33}(t)$ in Example 1.

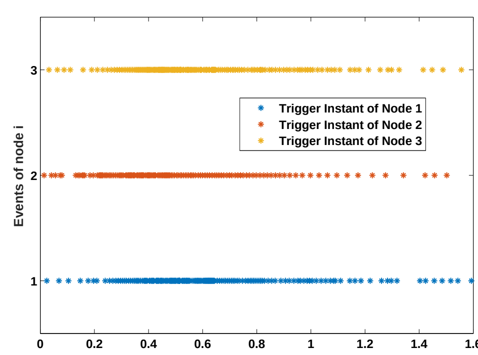

Fig. 8. Event-triggering times of node 1, node 2 and node 3 in Example 1.

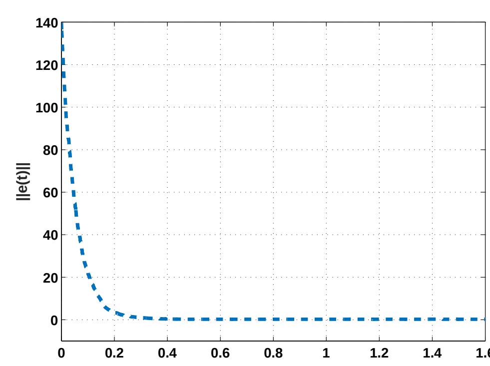

Fig. 9. Evolution of quasi-synchronization error $e(t)$ in Example 1.

a uniform distribution. Extensive simulations of distributed impulsive control were carried out in [24] but only on time-invariant heterogenous networks. The state trajectories of the virtual leader node and network nodes are plotted in Figs. 10-12, which show that quasi-synchronization can also be achieved by distributed impulsive control. From Fig. 13, we can see that the impulse happens at every time instant and the events of each node are triggered at every time instant, since there is no event-triggered strategy in distributed impulsive control. Indeed, the proposed event-trigger control is asynchronous control, while the distributed impulsive control is synchronous control which requires time synchronization among nodes. Fig. 14 shows that the nodes in switching network and the virtual leader are quasi-synchronized within the prescribed error bound when $t > 0.6$, which indicates a slower convergence rate than the proposed control method.

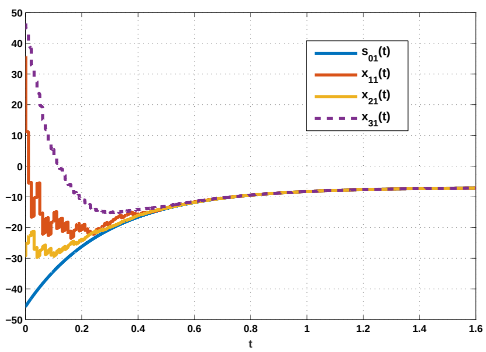

Fig. 10. State trajectories of $s_{01}(t)$, $x_{11}(t)$, $x_{21}(t)$, $x_{31}(t)$ via distributed impulsive control.

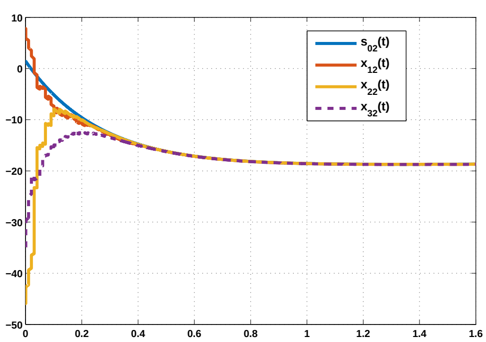

Fig. 11. State trajectories of $s_{02}(t)$, $x_{12}(t)$, $x_{22}(t)$, $x_{32}(t)$ via distributed impulsive control.

Example 2. *We compare the proposed method versus the event-triggered method [51] on a homogeneous Markovian switching network. Suppose the homogeneous network consists of three identical nodes described by* (1)*, where* $B_i = B = 0$*,* $x_i(t) = [x_{i1}(t), x_{i2}(t), x_{i3}(t)]^T$*,* $f(x_i(t)) = \sin(x_i)$*,* $i = 1, 2, 3$*. The dynamics of the virtual leader* $s(t) = [s_{01}(t), s_{02}(t), s_{03}(t)]^T$ *satisfies that* $\dot{s}(t) = f(s(t))$*.*

In this numerical example, the setting of simulation (the switching graph topologies, pinning nodes, pinning feedback gains, and initial values) is identical to that of Example 1. The results of simulation are depicted in Figs. 15-17. The state trajectories of the virtual leader node and network nodes are present in Fig. 15, which shows that the homogeneous Markovian switching network can be completely synchronized by the proposed design method. In Fig. 16, the events of each node under the proposed event-triggered approach are marked in the time interval $[0, 0.8]$. We can see that the sampling is sporadic rather than every time instant. Finally, Fig. 17 presents the evolution of the corresponding simulated errors in

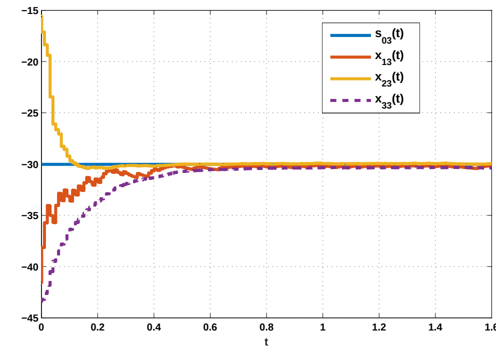

Fig. 12. State trajectories of $s_{03}(t)$, $x_{13}(t)$, $x_{23}(t)$, $x_{33}(t)$ via distributed impulsive control.

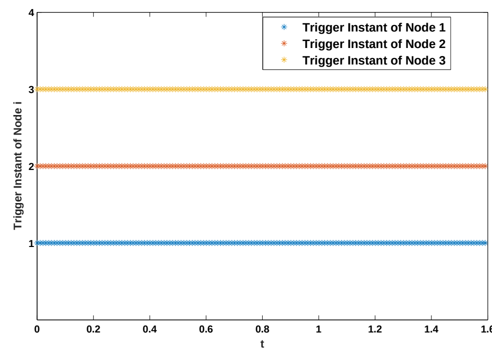

Fig. 13. Event-triggering times of node 1, node 2 and node 3 via distributed impulsive control.

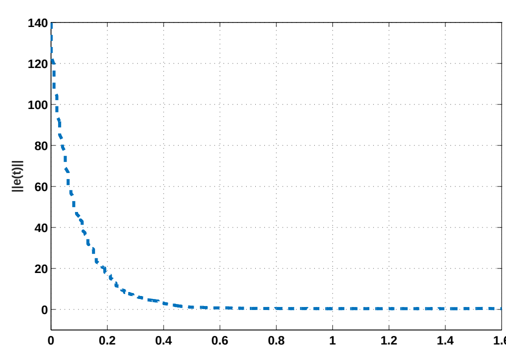

Fig. 14. Evolution of quasi-synchronization error $e(t)$ via distributed impulsive control.

Example 2. The red dot line denotes the evolution of $\|e(t)\|$ by the event-triggered method of [51] and the cyan dot line denotes the evolution of $\|e(t)\|$ by the proposed method. It can be observed that a faster convergence rate is yielded by the proposed method.

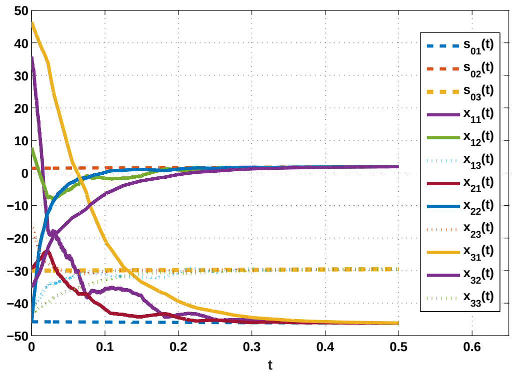

Fig. 15. State trajectories of $s(t) = [s_{01}(t), s_{02}(t), s_{03}(t)]^T$, $x_i(t) = [x_{i1}(t), x_{i2}(t), x_{i3}(t)]^T$, $i = 1, 2, 3$ in Example 2.

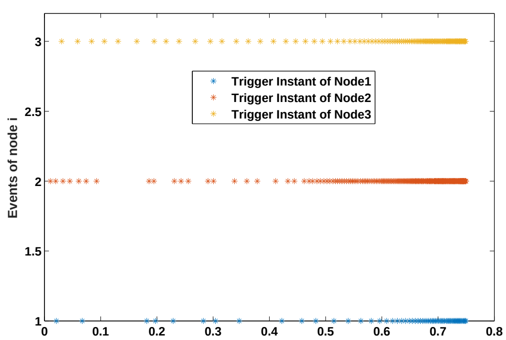

Fig. 16. Event-triggering times of node 1, node 2 and node 3 in Example 2.

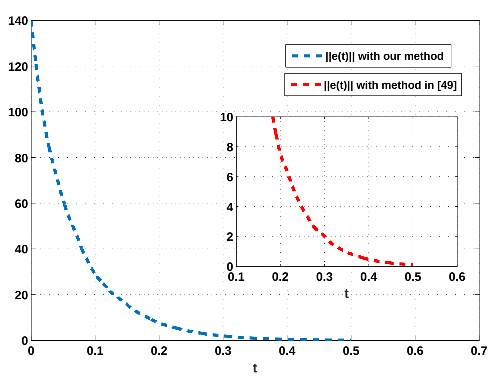

Fig. 17. Evolution of quasi-synchronization error $e(t)$ in Example 2.

V. Conclusion

In this paper, quasi-synchronization in a generalized Markovian switching heterogeneous network via stochastic sampling and event-triggered control was studied. First, we proposed a pinning strategy algorithm to determine how many and which nodes should be pinned in a generalized Markovian switching heterogeneous network. For the case where a network has limited control efficiency to get feasible pinning feedback gains, a condition has been provided to choose the coupling strength and the inner coupling matrix. Based on stochastic Lyapunov-Krasovskii stability theory and matrix inequalities technique, sufficient conditions giving an explicit expression of the quasi-synchronization error bound have been derived such that heterogeneous networks with generalized Markovian switching topologies can achieve quasi-synchronization exponentially under our stochastic event-triggering mechanism. The problems of achieving event-triggered output synchronization and designing distributed pinning schemes for switching heterogeneous networks will be investigated in our future studies.

Appendix A
Proof of Corollary 1

Proof. Choose the following Lyapunov function

$$V(t) = \frac{1}{2}\sum_{i=1}^{N} e_i^T(t)e_i(t).$$

The derivative of $V(t)$ along the trajectories of (12) can be obtained as follows

$$\begin{aligned}
\dot{V}(t) &= \sum_{i=1}^{N} e_i^T(t)Be_i(t) + \sum_{i=1}^{N} e_i^T(t)f_e(e_i(t), t) \\
&\quad - c\sum_{i=1}^{N}\sum_{j=1}^{N} l_{ij}(u)e_i^T(t)\Gamma e_j(t) - c\sum_{i=1}^{N} a_{i0}(u)e_i^T(t)\Gamma e_i(t) \\
&= \sum_{i=1}^{N} e_i^T(t)\,\mathrm{He}(\frac{1}{2}B)e_i(t) + \sum_{i=1}^{N} e_i^T(t)f_e(e_i(t), t) \\
&\quad - c\sum_{j=1}^{n} \alpha_j \hat{e}_j^T(t)\,\mathrm{He}(\frac{1}{2}L(u))\hat{e}_j(t) - c\sum_{j=1}^{n}\alpha_j\hat{e}_j^T(t)A_0(u)\hat{e}_j(t) \\
&\le \sum_{i=1}^{N} e_i^T(t)\,\mathrm{He}(\frac{1}{2}B)e_i(t) + \sum_{i=1}^{N} \bar{\gamma}e_i^T(t)e_i(t) \\
&\quad - c\sum_{j=1}^{n} \alpha_j \hat{e}_j^T(t)\,\mathrm{He}(\frac{1}{2}L(u))\hat{e}_j(t) - c\sum_{j=1}^{n}\alpha_j\hat{e}_j^T(t)A_0(u)\hat{e}_j(t) \\
&\le \sum_{j=1}^{n} \hat{e}_j^T(t)\Big\{(\lambda_{\max}\{\frac{1}{2}\,\mathrm{He}(B)\} + \bar{\gamma})I_N \\
&\quad - c\alpha_j[\frac{1}{2}\,\mathrm{He}(L(u)) + A_0(u)]\Big\}\hat{e}_j(t),
\end{aligned}$$

where $\hat{e}_j(t) = [e_1^j(t), e_2^j(t), \cdots, e_N^j(t)]^T$, $\sum_{i=1}^N e_i^T(t)e_i(t) = \sum_{j=1}^n \hat{e}_j^T(t)\hat{e}_j(t)$, and $e_1^j(t), e_2^j(t), \cdots, e_N^j(t)$ denote the jth element of the column vectors $e_1(t), e_2(t), \cdots, e_N(t)$, respectively.

According to (13), we have $\dot{V}(t) < 0$ and it implies that the error network (12) under limited control efficiency can be stable. This completes the proof. □

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Xinghua Liu (M'16) received the B.Sc. from Jilin University, Changchun, China, in 2009; and the Ph.D. degree in Automation from University of Science and Technology of China, Hefei, in 2014. From 2014 to 2015, he was invited as a visiting fellow at RMIT University in Melbourne, Australia. From 2015 to 2018, He was a Research Fellow at the School of Electrical and Electronic Engineering in Nanyang Technological University, Singapore. Dr. Liu has joined Xi'an University of Technology as a professor since September 2018. His research interest includes state estimation and control, intelligent systems, autonomous vehicles, cyber-physical systems, robotic systems, etc.

Wee Peng Tay (S'06 M'08 SM'14) received the B.S. degree in Electrical Engineering and Mathematics, and the M.S. degree in Electrical Engineering from Stanford University, Stanford, CA, USA, in 2002. He received the Ph.D. degree in Electrical Engineering and Computer Science from the Massachusetts Institute of Technology, Cambridge, MA, USA, in 2008. He is currently an Associate Professor in the School of Electrical and Electronic Engineering at Nanyang Technological University, Singapore. His research interests include information and signal processing over networks, distributed inference and estimation, information privacy, machine learning, information theory, and applied probability. Dr. Tay received the Tan Chin Tuan Exchange Fellowship in 2015. He is a coauthor of the best student paper award at the Asilomar conference on Signals, Systems, and Computers in 2012, and the IEEE Signal Processing Society Young Author Best Paper Award in 2016. He is currently an Associate Editor for the IEEE Transactions on Signal Processing, an Editor for the IEEE Transactions on Wireless Communications, and serves on the MLSP TC of the IEEE Signal Processing Society. He has also served as a technical program committee member for various international conferences.

Zhi-Wei Liu (M'14) received the B.S. degree in information management and information system from Southwest Jiaotong University, Chengdu, China, in 2004, and the Ph.D. degree in control science and engineering from the Huazhong University of Science and Technology, Wuhan, China, in 2011. He is currently an Associate Professor with the College of Automation, Huazhong University of Science and Technology. His current research interests include cooperative control and optimization of distributed network systems.

Gaoxi Xiao (M'99) received the B.S. and M.S. degrees in applied mathematics from Xidian University, Xiąŕan, China, in 1991 and 1994 respectively. He was an Assistant Lecturer in Xidian University in 1994-1995. In 1998, he received the Ph.D. degree in computing from the Hong Kong Polytechnic University. He was a Postdoctoral Research Fellow in Polytechnic University, Brooklyn, New York in 1999; and a Visiting Scientist in the University of Texas at Dallas in 1999-2001. He joined the School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore, in 2001, where he is now an Associate Professor. His research interests include complex systems and complex networks, communication networks, smart grids, and system resilience and risk management. Dr. Xiao serves/served as an Editor or Guest Editor for IEEE Transactions on Network Science and Engineering, PLOS ONE and Advances in Complex Systems etc., and a TPC member for numerous conferences including IEEE ICC and IEEE GLOBECOM etc.