MH7009 Homework 1

AY25/26 Semester 2

due in class on Tuesday, 10th February

Problem 1 (3 marks)**.** Let $X_1, \dots, X_n$ be i.i.d. random variables satisfying $\mathbb{E}X_i = 0$ and $\mathbb{E}X_i^2 = 1$. Show that $Z_n = (X_1 + \cdots + X_n)/\sqrt{n}$ does not converge in probability.

Problem 2 (3 marks)**.** Show that the convergence rate of $1/\sqrt{n}$ in Berry-Esseen Theorem cannot be improved.

Problem 3 (General Hoeffding's Inequality, 3 marks)**.** Let $X_1, \dots, X_n$ be independent random variables such that $\mathbb{E}X_i = 0$ and $X_i \in [c_i, d_i]$ almost surely for all i. Then

$$\Pr\left\{\left|\sum_{i=1}^{n} X_i\right| > t\right\} \leq 2\exp\left(-\frac{2t^2}{\sum_{i=1}^{n}(d_i - c_i)^2}\right), \quad t > 0.$$

Problem 4 (Mean estimation, 3 marks)**.** Suppose that $X_1, \dots, X_N$ are independent samples from a distribution with an unknown mean μ and known variance σ^2. Our goal is to obtain an estimate $\hat{\mu}$ such that $|\hat{\mu} - \mu| < \epsilon$. Show that, using $N = O(\sigma^2/\epsilon^2 \cdot \ln(1/\delta))$ samples, we can find a desirable $\hat{\mu}$ with probability at least $1 - \delta$ (where the probability is taken over the N samples).

Problem 5 (3 marks)**.** Show that under the assumption of $\mathbb{E}X = 0$, property (4) in the subgaussian lemma is equivalent to (1)–(3), and show that without the assumption of $\mathbb{E}X = 0$, property (4) in the subgaussian lemma may fail even if (1)–(3) hold.

Problem 6 (Maximum of subgaussians, 3 marks)**.** Let $X_1, X_2, \dots$ be a sequence of subgaussian random variables, which are not necessarily independent. Let $K = \max_i \|X_i\|_{\psi_2}$. Show that

$$\mathbb{E} \max_{1 \leq i \leq N} \frac{|X_i|}{\sqrt{1 + \ln i}} \leq C_1 K$$

for some absolute constant $C_1 > 0$. Deduce that for every $N \geq 2$ we have

$$\mathbb{E} \max_{1 \leq i \leq N} |X_i| \leq C_2 K \sqrt{\ln N}$$

for some absolute constant $C_2 > 0$.

Problem 7 (5 marks)**.** Show that the upper bound on the maximum in the preceding problem characterizes the subgaussian variables. That is, if $X_1, X_2, \dots$ are independent copies of some random variable X and it holds that

$$\mathbb{E} \sup_{1 \leq i \leq n} |X_i| \leq K\sqrt{\ln n}$$

for all $n \geq 2$, then X is subgaussian and $\|X\|_{\Psi_2} \leq CK$ for some absolute constant $C > 0$.

Problem 8 (3 marks)**.** Show that X is subgaussian if and only if X^2 is subexponential and $\|X\|_{\Psi_2}^2 \leq \|X^2\|_{\Psi_1} \leq 2\|X\|_{\Psi_2}^2$.

Problem 9 (5 marks)**.** Let $X_1, \dots, X_n$ be i.i.d. centred random variables such that

$$\Pr\{|X_i| > t\} \leq e^{-t^p}$$

for some $p \in [1,2]$. Let $q = p/(p-1)$ be the conjugate index of p. Let $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ and $Z = \sum_{i=1}^n a_i X_i$. Show that

$$\Pr\{Z \geq t\} \leq c_1 \exp\left(-c_2 \min\left\{\frac{t^2}{\|a\|_2^2}, \frac{t^p}{\|a\|_q^p}\right\}\right),$$

where c_1 is an absolute constant and c_2 is a constant that depends only on p. (Hint: Show that $\mathbb{E}\exp(\lambda X_i) \leq \exp(K(\lambda^2 + |\lambda|^q))$ for some K that depends only on p.)

Problem 10 (3 marks)**.** Let X be a subgaussian random variable such that $X \geq 0$ almost surely. Then

$$\mathbb{E}X \leq (\mathbb{E}|X|^p)^{1/p} \leq \mathbb{E}X + CK\sqrt{p}, \quad p \geq 1,$$

where $K = \|X - \mathbb{E}X\|_{\psi_2}$ and $C > 0$ is an absolute constant.

Problem 11 (3 marks)**.** Suppose that (T, d) is a metric space endowed by a probability measure μ. Let X be a random vector drawn from T according to the probability measure μ. Assume that there exists $K > 0$ such that

$$\|f(X) - \mathbb{E}f(X)\|_{\Psi_2} \leq K \|f\|_{Lip}$$

for every Lipschitz function $f : T \to \mathbb{R}$. Show that if $\mu(A) \geq 1/2$ then

$$\mu(A_t) \geq 1 - 2\exp\left(-\frac{ct^2}{K^2}\right), \quad t \geq 0,$$

where $c > 0$ is an absolute constant. (Hint: First show that we can replace the expectation with median with a constant factor larger K. Then consider $f(x) = d(x, A)$.)

Problem 12 (3 marks)**.** Let $X = (X_1, \ldots, X_n)$ be a vector of n random coordinates that satisfy $\mathbb{E}X_i = 0$, $\mathbb{E}X_i^2 = 1$ and $\|X_i\|_{\psi_2} \leq C$ for all i. Note that X_i is not necessarily Gaussian. Show that

$$\Pr\{(1-\epsilon)\sqrt{n} \leq \|X\|_2 \leq (1+\epsilon)\sqrt{n}\} \geq 1 - 2\exp(-c\epsilon^2 n), \quad \epsilon > 0.$$