MH7009 Homework 2

AY 25/26 Semester 2

due in class on Tuesday, 10th March

Problem 1 (3 marks). Show that $A = \frac{1}{\sqrt{k}} G$ works for Johnson-Lindenstrauss Lemma, where G is a random matrix with i.i.d. subgaussian entries such that $\mathbb{E} G_{ij} = 0$, $\mathbb{E} G_{ij}^2 = 1$ and $\|G_{ij}\|_{\psi_2} \le K$, where K is an absolute constant.

Problem 2 (3 marks). Show that ϕ defined in the lecture for the push-forward of the Gaussian measure on $\mathbb{R}^n$ to the uniform distribution on $[0,1]^n$ is Lipschitz and $\|\phi\|_{\mathrm{Lip}}$ is at most an absolute constant. Prove a Lipschitz concentration inequality on $[0,1]^n$ by using the Gaussian concentration to control the deviation of $f(\phi(g))$ in terms of $\|f \circ \phi\|_{\mathrm{Lip}} \le \|f\|_{\mathrm{Lip}} \|\phi\|_{\mathrm{Lip}}$.

Problem 3 (5 marks). Prove a Lipschitz concentration inequality for $B(0, \sqrt{n})$ using a similar idea of push-forward measure. (Hint: find a Lipschitz function $\phi : \mathbb{R}^n \to B(0, \sqrt{n})$ such that $\phi(g)$ is uniform in the ball $B(0, \sqrt{n})$ for $g \sim N(0, I_n)$.)

Problem 4 (3 marks). Show that X is isotropic iff $\mathbb{E}\langle X, x\rangle^2 = \|x\|_2^2$.

Problem 5 (3 marks). Show that if a vector X has subgaussian coordinates then X is subgaussian. However, it is possible that $\|X\|_{\Psi_2} \gg \max_i \|X_i\|_{\Psi_2}$.

Problem 6 (5 marks). Let X be an isotropic random vector supported on a finite set $T \subset \mathbb{R}^n$. Further suppose that X is subgaussian with $\|X\|_{\psi_2} \le C$ for some absolute constant C. Then it must hold that

$$|T| \ge e^{cn}$$

for some absolute constant $c > 0$. (Hint: suppose that $|T| = N$ and $\|X\|_{\psi_2} \le C$. Construct N subgaussian random variables $X_1, \ldots, X_N$ (not necessarily independent) such that $\max_i \|X_i\|_{\psi_2} \le C'$ and $\mathbb{E} \max_i |X_i| \ge c\sqrt{n}$ for some absolute constants $C', c > 0$. Use the upper bound for the expected maximum of subgaussian variables to derive a lower bound for N.)

Problem 7 (3 marks). Let B_1^n denote the unit ℓ_1 ball in $\mathbb{R}^n$, that is, $B_1^n = \{x \in \mathbb{R}^n : \|x\|_1 \le 1\}$. Show that there exists a constant $c > 0$ such that the dilated ℓ_1 ball cB_1^n is an isotropic convex body but the uniform distribution on cB_1^n is not subgaussian (the subgaussian norm is not an absolute constant).

Problem 8 (3 marks). An alternative way to define a subgaussian vector is as follows. For a random vector X, define

$$\Lambda_X(\lambda) = \sup_f \mathbb{E} e^{\lambda(f(X) - \mathbb{E} f(X))},$$

where the supremum is taken over all 1-Lipschitz functions $f : \mathbb{R}^n \to \mathbb{R}$. We say X is subgaussian if $\Lambda_X(\lambda) \le e^{\lambda^2 K^2}$ for all $\lambda \in \mathbb{R}$, where K is some constant. We define the smallest such K to be the subgaussian norm of X.

Compare this definition with our subgaussian norm defined via one-dimensional marginal distributions. Are they equivalent (differing by an absolute constant)? If not, which one is larger and by how much is it larger?

Problem 9 (3 marks)**.** Prove the theorem of covariance estimation with general Σ.

Problem 10 (3 marks)**.** Show that the $\sqrt{\ln n}$ factor is necessary in non-commutative Khinchine inequality (NCKI) for the operator norm. Note that ϵ_i's are Rademacher variables. (Hint: it may be useful to use the fact that $\max_i |g_i| \gtrsim \sqrt{\ln n}$ for i.i.d. $g_1, \ldots, g_n \sim N(0,1)$).

Problem 11 (3 marks)**.** Show the following Bernstein-type inequality. Let $A_1, \ldots, A_N$ be (deterministic) symmetric matrices and $\epsilon_1, \ldots, \epsilon_N$ be a Rademacher sequence. Let $\sigma^2 = \left\| \sum_{i=1}^N A_i^2 \right\|_{op}$. Show the following tail abound and deduce the NCKI for the operator norm from it.

$$\Pr\left\{ \left\| \sum_i \epsilon_i A_i \right\|_{op} \geq t \right\} \leq 2ne^{-t^2/(2\sigma^2)}, \quad t > 0.$$

Problem 12 (3 marks)**.** Under the same setting of Proposition 37 of the lecture notes (the symmetrization technique), prove that the Rademacher sum is dominated by the Gaussian sum:

$$\mathbb{E} \left\| \sum_i \epsilon_i X_i \right\| \leq C \mathbb{E} \left\| \sum_i g_i X_i \right\|,$$

where $C > 0$ is an absolute constant and $g_1, \ldots, g_n$ are i.i.d. $N(0,1)$ variables. What is the value of your constant C? Can you show that it is optimal? What can you say about the reverse, that is, can you find a quantity L as small as possible (which may depend on n but must not depend on the normed space) such that $\mathbb{E} \left\| \sum_i g_i X_i \right\| \leq L \mathbb{E} \left\| \sum_i \epsilon_i X_i \right\|$?