MH7009 Homework 3

AY25/26 Semester 2

due in class on Tuesday, 7th April

Problem 1 (Matrix moment inequality for PSD matrices, 3 marks)**.** Let $p \geq 1$ and $W_1, \ldots, W_m$ be independent, $n \times n$ symmetric PSD matrices. Show that

$$\left(\mathbb{E}\left\|\sum_i W_i\right\|_{op}^p\right)^{1/p} \leq C\left[\left\|\sum_i \mathbb{E}\, W_i\right\|_{op} + \max\{p, \ln n\} \cdot \left(\mathbb{E}\max_i \|W_i\|_{op}^p\right)^{1/p}\right].$$

for some absolute constant $C > 0$.

Problem 2 (3 marks)**.** Show that the conclusion in Slepian's Lemma cannot be changed to $\mathbb{E}\sup_t |X_t| \leq \mathbb{E}\sup_t |Y_t|$, even when X and Y are independent. (Hint: There is a counterexample when $|T| = 2$, that is, X and Y are two-dimensional gaussian vectors.)

Problem 3 (3 marks)**.** Let $u, v, w, z \in \mathbb{S}^{n-1}$. Show that $\|uv^T - wz^T\|_F^2 \leq \|u - w\|_2^2 + \|v - z\|_2^2$.

Problem 4 (Gaussian orthogonal ensemble, 3 marks)**.** Let A be a symmetric random matrix, in which the above-diagonal entries are i.i.d. $N(0,1)$ and the diagonal entries are i.i.d. $N(0,2)$. The diagonal entries and the above-diagonal ones are independent. This distribution of such A is called the Gaussian orthogonal ensemble (GOE). Show that $\mathbb{E}\,\|A\|_{op} \leq 2\sqrt{n} + C$ for some absolute constant C.

Problem 5 (3 marks)**.** In this problem we verify (a weaker version of) the bullet point (4) in the remark after the NCKI for the operator norm.

Let A be a symmetric $n \times n$ random matrix ($n \geq 2$), in which the above-diagonal entries are i.i.d. Rademacher variables and the diagonal entries are zeroes. Our goal is to show that there exist absolute constants $C, c > 0$ such that $c\sqrt{n} \leq \|A\|_{op} \leq C\sqrt{n}$.

(i) Show that $\|A\|_{op} \geq (1 - o(1))\sqrt{n}$, where $o(1)$ represents a quantity which tends to 0 as $n \to \infty$.

(ii) Show that $\|A\|_{op} \leq C\sqrt{n}$ for some absolute constant C. Find the value of your C. Make C as small as possible.

Problem 6 (Dudley's integral: tail bound, 3 marks)**.** Let $\{X_t\}_{t\in T}$ be as in the theorem of Dudley's integral in class. Show that it holds with probability at least $1 - C_1\exp(-u^2)$ that

$$\sup_{t,s\in T} |X_t - X_s| \leq C_2\left[\int_0^\infty \sqrt{\ln N(T, d, \epsilon)}d\epsilon + u \cdot \operatorname{diam}(T)\right],$$

where $C_1, C_2 > 0$ are absolute constants.

Problem 7 (Generalized Dudley's integral, 5 marks)**.** Follow the approach in class and prove the following generalized Dudley's integral.

Let $\psi : [0, \infty) \to \mathbb{R}$ be an increasing convex function such that

(i) $\psi(0) = 0$;

(ii) $\lim_{x\to+\infty} \psi(x) = +\infty$;

(iii) $\psi^{-1}(xy) \leq C(\psi^{-1}(x) + \psi^{-1}(y))$ for $x, y \geq 1$ and some constant C;

(iv) $\int_0^a \psi^{-1}(1/x)dx < \infty$ for some $a > 0$,

where ψ^{-1} is the inverse function of ψ. For a random variable X, define its ψ-norm to be

$$\|X\|_\psi = \inf\{c : \mathbb{E}\,\psi(|X|/c) \leq 1\}.$$

Let (T, d) be a compact metric space. Suppose that $\{X_t\}_{t\in T}$ is a *centred* stochastic process indexed by $t \in T$ with incremental condition

$$\|X_s - X_t\|_\psi \leq d(s, t),$$

for any $s, t \in T$. Prove that

$$\mathbb{E} \sup_{t\in T} X_t \leq L \int_0^{\mathrm{diam}(T)} \psi^{-1}(N(T, d, \epsilon))d\epsilon.$$

for some absolute constant $L > 0$.

Problem 8 (3 marks)**.** Show that when T is finite and $|T| \geq T_0$ for some absolute constant T_0, it holds that

$$\sum_{k\geq 0} 2^{k/2} e_k(T) \lesssim \gamma_2(T, d) \cdot \log\log |T|.$$

Problem 9 (3 marks)**.** Show that when $T \subseteq \mathbb{R}^n$ and the metric d on T is induced by a norm, that is, $d(x, y) = \|x - y\|$ for some norm $\|\cdot\|$, it holds that

$$\sum_{k\geq 0} 2^{k/2} e_k(T) \lesssim \gamma_2(T, d) \cdot \ln(n + 1).$$

(Hint: show that nothing interesting happens after $k \geq \log_2 n + C \log_2 \log_2 n$ for some (large) absolute constant $C > 0$.)

Problem 10 (3 marks)**.** Construct nets $T_0, T_1, \dots \subseteq T = B_{\ell_1}^n$ such that $|T_k| \leq 2^{2^k}$ and

$$\sup_{x\in B_{\ell_1}^n} \sum_{k=0}^{\infty} 2^{\frac{k}{2}} d(x, T_k) \leq C\sqrt{\log n}.$$

Problem 11 (Higher-dimensional Hanson-Wright, 3 marks)**.** Let $X_1, X_2, \dots, X_n$ be independent, mean zero, subgaussian random vectors in $\mathbb{R}^d$. Let $A = (a_{ij})$ be an $n \times n$ matrix with zero diagonal entries. Show that for $t \geq 0$,

$$\Pr\left\{\left|\sum_{\substack{1\leq i,j\leq n\\ i\neq j}} a_{ij}\langle X_i, X_j\rangle\right| > t\right\} \leq 2\exp\left(-c\min\left\{\frac{t^2}{K^4 d\|A\|_F^2}, \frac{t}{K^2\|A\|_{op}}\right\}\right),$$

where $K = \max_i \|X_i\|_{\Psi_2}$.