1 Concepts of Distributions

1.1 Suppose that $1 \le p < \infty$, show that $C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$.

Proof. Suppose that $u \in L^p(\Omega)$ and we can assume that Ω is bounded. (Otherwise take $\Omega_n = \Omega \cap B(0,n)$ and thus we can find $u_n = u\chi_{\Omega_n}$ for some n such that $\|u_n - u\|_p < \epsilon$, and we will approximate u_n on Ω_n) We are going to find $\psi \in C_0^\infty(\Omega)$ such that $\|u - \psi\|_p < 2\epsilon$ in two steps.

(1) Find $\phi \in C_0(\Omega)$ such that $\|u - \phi\|_p < \epsilon$.
Since u is in $L^p(\Omega)$ we can find u_1 in $L^p(\Omega)$ which is bounded (say, by M) and satisfies $\|u_1 - u\|_p < \epsilon/3$. Now we can choose $K \subseteq \Omega' \subseteq \Omega$, where K is closed and Ω' is open, such that $m(\Omega \setminus K) < (\frac{\epsilon}{3M})^p$ and $m(\Omega' \setminus K) < (\frac{\epsilon}{6M})^p$. Then $\|u_1\chi_K - u_1\|_p \le Mm(\Omega \setminus K)^{\frac{1}{p}} < \epsilon/3$. Finally, from Luzin's Theorem, we know that there exists $\phi \in C(\Omega)$ with support contained in $\overline{\Omega'}$ and bounded by M, such that $m(E) < (\frac{\epsilon}{12M})^p$, where $E = \{x : u_1\chi_K \ne \phi\}$. Thus $\|u_1\chi_K - \phi\|_p < 2M \cdot m(E)^{\frac{1}{p}} + M \cdot m(\Omega' \setminus K))^{\frac{1}{p}} < \epsilon/3$, which implies that $\|u - \phi\|_p < \epsilon$.

(2) Find $\psi \in C_0^\infty(\Omega)$ such that $\|\phi - \psi\|_p < \epsilon$.
Since as Ω is bounded and ϕ_δ converges to ϕ uniformly on Ω as $\delta \to 0^+$, we can just let $\psi = \phi_\delta$ for some appropriate δ. □

1.2 Prove that δ is not locally integrable.

Proof. Note that $e^x\delta = \delta$, hence if δ is locally integrable, we must have $e^x\delta = \delta$ a.e., yielding $\delta = 0$ a.e.. But δ is not a zero distribution, contradiction. Therefore δ cannot be locally integrable. □

1.3 Suppose that

$$f_j(x) = \left(1 + \frac{x}{j}\right)^j \quad (j = 1, 2, \dots)$$

Show that $f_j(x) \to e^x$ in $\mathscr{D}'(\mathbb{R})$.

Proof. For any $\phi \in \mathscr{D}(\mathbb{R})$ we have that $(1 + \frac{x}{j})^j\phi(x) \to e^x\phi(x)$ as $n \to \infty$ and $|(1 + \frac{x}{j})^j| \le e^{|x|}$ and $e^{|x|}|\phi(x)| \in L^1(\mathbb{R})$, hence by Lebesgue's Dominated Convergence Theorem it holds that

$$\lim_{j\to\infty} \langle f_j, \phi \rangle = \lim_{j\to\infty} \int_{\mathbb{R}} f_j(x)\phi(x)dx = \int_{\mathbb{R}} e^x\phi(x)dx = \langle e^x, \phi(x) \rangle.$$

and thus $f_j \to e^x$ weakly-star. □

1.4 Show that in $\mathscr{D}'(\mathbb{R})$,

(1) $\dfrac{1}{\pi}\dfrac{\epsilon}{x^2 + \epsilon^2} \to \delta(x)(\epsilon \to 0^+)$

(2) $\dfrac{1}{2\sqrt{\pi t}} \exp\left(-\dfrac{x^2}{4t}\right) \to \delta(x)(t \to 0^+)$

Proof. We prove a more general proposition that if nonnegative $f \in L^1$ with $\int_{\mathbb{R}} f(x)dx = 1$, then $f_\delta \to \delta$ weakly-star as $\delta \to 0^+$, where f_δ is defined by $f_\delta(x) = f(x/\delta)/\delta$. Item (a) is a special case of $f(x) = \frac{1}{\pi(1+x^2)}$ with $\delta = \epsilon$ and item (b) $f(x) = \frac{1}{2\sqrt{\pi}}e^{-x^2/4}$ with $\delta = \sqrt{t}$.

Since $\int_{\mathbb{R}} f(x)dx = 1$ we know that $\int_{\mathbb{R}} f_\delta(x)dx = 1$, hence for $\phi \in \mathscr{D}(\Omega)$ it holds that

$$|\langle f_\delta, \phi\rangle - \phi(0)| = \left|\int_{\mathbb{R}} f_\delta(x)\phi(x)dx - \phi(0)\right| = \left|\int_{\mathbb{R}} f_\delta(x)(\phi(x) - \phi(0))dx\right| \le \int_{\mathbb{R}} f_\delta(x)|\phi(x) - \phi(0)|dx$$

Since ϕ is continuous at $x = 0$ there exists δ_1 such that $|\phi(x) - \phi(0)| < \epsilon/2$ whenever $|x| < \delta_1$. Also since $f \in L^1(\mathbb{R})$, there exists δ_2 such that $\int_{|x|\geq 1/\delta_2} f < \epsilon/(2\|f\|_\infty)$. Let $\eta = \min\{\delta_1, \delta_2\}$. It follows that for $\delta < \eta$,

$$\begin{aligned}
|\langle f_\delta, \phi\rangle - \phi(0)| &\leq \int_{|x|\leq\delta} f_\delta(x)|\phi(x) - \phi(0)|dx + \int_{|x|>\delta} f_\delta(x)|\phi(x) - \phi(0)|dx \\
&\leq \frac{\epsilon}{2}\int_{|x|\leq\delta} f_\delta(x)dx + 2\|\phi\|_\infty \int_{|x|>\delta} f_\delta(x)dx \\
&\leq \frac{\epsilon}{2} + 2\|\phi\|_\infty \int_{|u|>\frac{1}{\delta}} f(u)du \\
&\leq \frac{\epsilon}{2} + 2\|\phi\|_\infty \int_{|u|>\frac{1}{\delta_2}} f(u)du < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Therefore $\langle f_\delta, \phi\rangle \to \phi(0)$ as $\delta \to 0^+$, or, $f_\delta \to \delta$ weakly-star. □

1.5 Let $\Omega \subseteq \mathbb{R}^n$ be an open set and K be compact subset of Ω. Show that there exists $\phi \in C_0^\infty(\Omega)$ such that $0 \leq \phi(x) \leq 1$ and $\phi(x) = 1$ in a neighbourhood of K.

Proof. Let $K_\delta = \{x : d(x, K) \leq \delta\}$ then $K_\delta \subseteq \Omega$ when δ is small enough. Then let

$$\psi(x) = \int_{K_\delta} j_{\frac{\delta}{2}}(y - x)dy.$$

It is clear that (a) $\psi \in C_0^\infty(\Omega)$ (since K_δ is bounded, differentiation can be performed under the integral sign); (b) $|\psi(x)| \leq 1$ for all $x \in \Omega$; and (c) $\psi(x) = 1$ for all $x \in B(K, \delta/2)$. □

2 The space of B_0

2.1 Verify that the convergence in $\mathscr{E}(\Omega)$ in Example 3.2.6 is independent of the choice of $\{K_m\}$.

Proof. Suppose that $\|\cdot\|_m$ are induced by $\{K_m\}$ and $\|\cdot\|'_m$ by $\{K'_m\}$. It suffices to show that for any m there exists m' and a constant C such that

$$\|\phi\|_m \leq C \cdot \|\phi\|'_{m'}, \quad \forall \phi \in \mathscr{E}(\Omega) \tag{1}$$

and for any m' there exists m' and a constant C' such that

$$\|\phi\|'_{m'} \leq C' \cdot \|\phi\|_m, \quad \forall \phi \in \mathscr{E}(\Omega). \tag{2}$$

We prove (1) here, and the proof of (2) is highly similar. It suffices to show that for any K_m it is contained in some $K_{m'}$. If not, there exists $x_i \in K_m$ such that $x_i \notin K'_{n_i}$ with $n_i \to \infty$ as $i \to \infty$. Since K_m is compact, $\{x_i\}$ has a convergent subsequence which goes to x. For simplicity, we assume that $x_i \to x$. Since $x \in \Omega = \bigcup_{m'=1}^\infty \mathrm{int}(K_{m'})$, we have m'_1 such that x is an interior point of $K_{m'_1}$. Thus x_n with n large enough are all contained in $K_{m'_1}$, and thus in K_{n_j} for j large enough. This is a contradiction with our choice of x_i. □

2.2 Let

$$\|\phi\|'_m = \sup_{\substack{|k|,|\alpha|\leq m \\ x\in\mathbb{R}^n}} |x^k \partial^\alpha \phi(x)|. \quad (m = 0, 1, 2, \dots)$$

Show that $\|\cdot\|'_m$ are equivalent countably many norms on $\mathscr{S}(\mathbb{R}^n)$.

Proof. Since $(1+|x|^2)^{\frac{m}{2}} \geq |x|^m$, we have that $\|\phi\|'_m \leq \|\phi\|_m$. On the other hand, denote $m' = \lceil m/2 \rceil$, then $m \leq 2m'$ and we have

$$\|\phi\|_m \leq \sum_{k=0}^{m'} \sup_{\substack{|\alpha|\leq m \\ x\in\mathbb{R}^n}} C_k |x|^{2k} |\partial^\alpha \phi(x)| \leq \sum_{k=0}^{m'} C_k \|\phi\|'_{2m'},$$

where C_k are constants. □

2.3 Show that $\mathscr{D}_K(\Omega)$ and $\mathscr{E}(\Omega)$ are both B_0 spaces.

Proof. Suppose that $\{\phi_k\}$ is Cauchy in $\mathscr{D}_K(\Omega)$ then it is a uniform Cauchy sequence, and thus is convergent to some function ϕ. It is clear that ϕ is continuous and has support in K. Also, $\{\partial^{(1,0,\cdots)}\phi_k\}$ is a Cauchy sequence and thus is convergent to some continuous function g. From the uniform convergence of $\{\partial^{(1,0,\cdots)}\phi_k\}$ it must hold that $\partial^{(1,0,\cdots)} f = g$. Therefore we know that $\phi \in \mathscr{D}_k(\Omega)$ and $\mathscr{D}_k(\Omega)$ is complete.

Now we show that $\mathscr{E}(\Omega)$ is complete. Suppose that $\{K_m\}$ is a sequence of increasing compact sets contained in Ω and $\Omega = \bigcup_{m=1}^{\infty} K_m$. Let $\{\phi_k\}$ be a Cauchy sequence in $\mathscr{E}(\Omega)$, then it is uniformly convergent on every K_m. Hence $\{\phi_k(x)\}$ is Cauchy for every x and thus $\{\phi_k\}$ is convergent to some ϕ pointwise. Similarly $\{\partial^{(1,0,\cdots)}\phi_k\}$ is convergent to some g. On every K_m the convergence is uniformly thus $f' = g$ on every K_m and thus for all $x \in \Omega$. Therefore we conclude that $\phi \in \mathscr{E}(\Omega)$ and $\mathscr{E}(\Omega)$ is complete. □

2.4 Suppose that $\mathscr{X}$ is a B_0 space, show X' is complete under weak-star convergence. In particular, $\mathscr{D}'_K$, S' and $\mathscr{E}'$ are complete.

Proof. Suppose that $\{f_n\}$ is a weak-star Cauchy sequence in X', that is, for any $x \in \mathscr{X}$, $\{f_n(x)\}$ is Cauchy. Thus the limit of $\{f_n(x)\}$ exists for every $x \in \mathscr{X}$, call it $f(x)$. In this way we define a functional f on $\mathscr{X}$ and it is clear that f is linear. Now we shall show that f is continuous, that is, $f(x_k) \to 0$ whenever $x_k \to 0$ in X.

Since $\{f_n(x)\}$ exists for all $x \in \mathscr{X}$, $\{f_n(x)\}$ is bounded. Notice that $\mathscr{X}$ is of second category (it is a Frechet space), we can apply Uniform Boundedness Principle that there exists $\{M_k\}$ such that $|f_n(x)| \leq M_k\|x\|_k$ for each k and therefore $|f(x)| \leq M_k\|x\|_k$. The conclusion follows easily. □

2.5 Let G be a bounded open simply-connected region on the complex plane. Denote by $A(G)$ all the analytic functions over G and define a family of seminorms as follows. Let

$$G_1 \subset \overline{G_1} \subset G_2 \subset \overline{G_2} \subset \cdots \subset G_m \subset \overline{G_m} \subset \cdots \subset G$$

is a sequence of connected sets, where G_m $(m = 1, 2, \dots)$ is open and its boundary consists of finitely many curves with finite length. Also $\bigcup_{i=1}^{m} \overline{G_m} = G$. Let

$$\|\phi\|_m = \max_{z\in\overline{G_m}} |\phi(z)|, \quad \forall\phi \in A(G).$$

Show that $A(G)$ is a B_0 space. Suppose that $\{\phi_n\} \subset A(G)$ and there exists $\{M_n\}$ such that

$$\|\phi_n\|_m \leq M_m \quad (m = 1, 2, \dots; n = 1, 2, \dots)$$

then $\{\phi_n\}$ must have a convergent subsequence.

Proof. Obviously $A(G)$ is a B_0^* space. Since $\bigcup_m G_m = G$, from a similar argument in Problem 1, we know that each compact set $K \subset G$ is contained in some $\overline{G_m}$. Hence if we want to prove that some property holds for any compact set in G, it suffices to show the property holds for all $\overline{G_m}$.

Suppose that $\{\phi_k\}$ is a Cauchy sequence, then ϕ_k is uniformly convergent on G_m, thus $\phi_k \to \psi_m$ for some ψ_m on K_m. Since $\{\phi_k\}$ are analytic in G_m, ψ_m is analytic in G_m. Also it is easy to see that those $\{\psi_m\}$ actually coincides, and thus a function ψ, which is analytic in G, is well-defined, and $\phi_k \to \psi$ in $A(G)$.

Now suppose that $\|\phi_n\|_m \leq M_m$ for all m, we shall show that $\{\phi_n\}$ is equicontinuous on $\overline{G_m}$. Let C be the boundary of a closed disc in G_m of radius r. If z, z_0 are inside G_m then by Cauchy's integral theorem we obtain that

$$\phi_n(z) - \phi_n(z_0) = \frac{1}{2\pi i}\int_C \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - z_0}\right)\phi_n(z)dz = \frac{z - z_0}{2\pi i}\int_C \frac{\phi_n(\zeta)d\zeta}{(\zeta - z)(\zeta - z_0)}.$$

If $|\phi_n(z)| \leq M$ on C, we restrict z and z_0 to the smaller concentric disc of radius $r/2$ and obtain that

$$|\phi_n(z) - \phi_n(z_0)| \leq \frac{4M_m|z - z_0|}{r},$$

which shows the equicontinuity on the smaller disc. Now it is easy to take the approach of choosing a finite subcovering from a covering of $\overline{G_m}$, proving that $\{\phi_n\}$ is equicontinuous on $\overline{G_m}$. The conclusion follows from an obvious diagonalisation argument. □

3 Operations on Distributions

3.1 Calculate

(1) $\tilde{\partial}_x^n|x|$;

(2) $\tilde{\partial}^n x_+^\lambda$ ($\lambda \in \mathbb{R}, \lambda \neq -1, -2, \dots$), where

$$x_+^\lambda = \begin{cases} x^\lambda, & x > 0. \\ 0, & x \leq 0. \end{cases}$$

Proof. (1) Assume $n \geq 1$. Let $\phi \in \mathscr{D}(\mathbb{R})$, then

$$\begin{aligned}\langle \tilde{\partial}_x^n|x|, \phi\rangle = (-1)^n\langle |x|, \partial^n\phi\rangle &= (-1)^n\left(\int_0^\infty x\partial^n\phi(x)dx - \int_{-\infty}^0 x\partial^n\phi(x)dx\right) \\ &= (-1)^n\left(-\int_0^\infty \phi^{(n-1)}(x)dx + \int_{-\infty}^0 \phi^{(n-1)}(x)dx\right)\end{aligned}$$

If $n = 1$ then we find that $\langle \tilde{\partial}_x^n|x|, \phi\rangle = \langle \operatorname{sgn} x, \phi\rangle$. If $n = 2$, we proceed as

$$\begin{aligned}\langle \tilde{\partial}_x^n|x|, \phi\rangle &= (-1)^n(-(0 - \phi^{(n-2)}(0)) + \phi^{(n-2)}(0) - 0)) \\ &= 2(-1)^{n-2}\phi^{(n-2)}(0) \\ &= 2\langle \delta^{(n-2)}, \phi\rangle\end{aligned}$$

Therefore, we conclude that

$$\partial_x^n|x| = \begin{cases} \text{sgn}, & n = 1; \\ 2\delta^{(n-2)}, & n \geq 2. \end{cases}$$

(2) Let $\phi \in \mathscr{D}(\mathbb{R})$, then for $\lambda > -1$ we have

$$\langle x_+^\lambda, \phi\rangle = \int_0^\infty x^\lambda\phi(x)dx$$

well-defined, and we can rewrite it as

$$\langle x_+^\lambda, \phi\rangle = \frac{(-1)^k}{(\lambda+1)(\lambda+2)\cdots(\lambda+k)}\int_0^\infty x^{\lambda+k}\phi^{(k)}(x)dx,$$

which is well-defined for $\lambda \in (-k+1, -k)$. It is also well-defined for all $\lambda > -(k+1)$ except negative integers. Then it is easy to see that

$$\langle \tilde{\partial}_x^n x_+^\lambda, \phi\rangle = (-1)^n\langle x_+^\lambda, \phi^{(n)}\rangle = (\lambda - n + 1)\cdots\lambda\langle x_+^{\lambda-n}, \phi\rangle.$$

Hence

$$\tilde{\partial}^n x_+^\lambda = \lambda(\lambda-1)\cdots(\lambda-(n-1))x_+^{\lambda-n}.$$ □

3.2 Show that

$$\frac{\tilde{d}}{dx}\ln|x| = \mathrm{pv}\,\frac{1}{x},$$

i.e.,

$$\left\langle \frac{\tilde{d}}{dx}\ln|x|, \phi \right\rangle = \lim_{\epsilon\to 0^+}\int_{|x|\ge\epsilon}\frac{\phi(x)}{x}dx, \quad \forall\phi\in\mathscr{D}(\mathbb{R}).$$

Proof. This is very straight-forward. Let $\phi\in\mathscr{D}(\mathbb{R})$ then

$$\begin{aligned}
\langle\frac{\tilde{d}}{dx}\ln|x|, \phi\rangle = -\langle \ln|x|, \phi'\rangle &= -\int_{\mathbb{R}}\ln|x|\phi'(x)dx \\
&= -\lim_{\epsilon\to 0^+}\left(\int_\epsilon^\infty \phi'(x)\ln x dx + \int_{-\infty}^{-\epsilon}\phi'(x)\ln(-x)dx\right) \\
&= -\lim_{\epsilon\to 0^+}\left(-\phi(\epsilon)\ln\epsilon - \int_\epsilon^\infty\frac{\phi(x)}{x}dx + \phi(-\epsilon)\ln\epsilon - \int_{-\infty}^{-\epsilon}\frac{\phi(x)}{x}dx\right) \\
&= \lim_{\epsilon\to 0^+}\left((\phi(\epsilon)-\phi(\epsilon))\ln\epsilon + \int_\epsilon^\infty\frac{\phi(x)}{x}dx + \int_{-\infty}^{-\epsilon}\frac{\phi(x)}{x}dx\right)
\end{aligned}$$

Note that $(\phi(\epsilon)-\phi(\epsilon))\ln\epsilon = 2\epsilon\phi'(\epsilon)\ln\epsilon \to 0$ as $\epsilon\to 0^+$ since ϕ' is bounded. It follows that

$$\langle\frac{\tilde{d}}{dx}\ln|x|, \phi\rangle = \lim_{\epsilon\to 0^+}\int_{|x|\ge\epsilon}\frac{\phi(x)}{x}dx.$$

□

3.3 Suppose that $\Omega = (a,b)\subset\mathbb{R}$, $x_0\in\Omega$ and $f\in C^1(\Omega\setminus\{x_0\})$ with the discontinuity of the first kind at x_0. Also suppose that f' is bounded in $\Omega\setminus\{x_0\}$. Show that

$$\frac{\tilde{d}}{dx}f = f' + (f(x_0^+) - f(x_0^-))\delta(x_0).$$

Proof. Let $\phi\in\mathscr{D}(\Omega)$,

$$\begin{aligned}
\left\langle\frac{\tilde{d}}{dx}f, \phi\right\rangle = -\langle f, \phi'\rangle &= -\int_a^b f(x)\phi'(x)dx \\
&= -\left(\int_a^{x_0} f(x)\phi'(x)dx + \int_{x_0}^b f(x)\phi'(x)dx\right) \\
&= -\left(f(x)\phi(x)|_a^{x_0^-} - \int_a^{x_0} f'(x)\phi(x)dx + f(x)\phi(x)|_{x_0^+}^b - \int_{x_0}^b f'(x)\phi(x)dx\right) \\
&= \phi(x_0)(f(x_0^+) - f(x_0^-)) + \int_a^b f'(x)\phi(x)dx \\
&= (f(x_0^+) - f(x_0^-))\langle\delta(x_0), \phi\rangle + \langle f', \phi\rangle
\end{aligned}$$

□

3.4 Prove that for all $f\in\mathscr{D}'(\mathbb{R}^n)$ it holds that

$$\tilde{\partial}_{x_i}f = \lim_{h\to 0}\frac{1}{h}(\tilde{\tau}_{-he_i}f - f),$$

where

$$e_i = (\underbrace{0,\dots,0,1}_{i},0,\dots,0) \quad (i = 1,2,\dots,n).$$

Proof. Let $\phi \in \mathscr{D}(\mathbb{R}^n)$. We shall prove that $\{(\tau_{-he_i}\phi - \phi)/h\}$ converges to $\partial_{x_i}\phi$ in $\mathscr{D}(\mathbb{R}^n)$ as $h \to 0$, afterwards we would have

$$\begin{aligned}
\langle \tilde{\partial}_{x_i} f, \phi \rangle = -\langle f, \partial_{x_i}\phi \rangle &= -\left\langle f, \lim_{h\to 0} \frac{1}{h}(\tau_{-he_i}\phi - \phi) \right\rangle \\
&= -\lim_{h\to 0} \left\langle f, \frac{1}{h}(\tilde{\tau}_{-he_i}\phi - \phi) \right\rangle \\
&= -\lim_{h\to 0} \frac{1}{h}\left(\langle \tilde{\tau}_{he_i} f, \phi \rangle - \langle f, \phi \rangle \right), \\
&= \lim_{h\to 0} \frac{1}{h'}\left(\langle \tilde{\tau}_{-h'e_i} f, \phi \rangle - \langle f, \phi \rangle \right), \quad (\text{let } h' = -h)
\end{aligned}$$

which is desired. To show that $\{(\tau_{-he_i}\phi - \phi)/h\}$ converges to $\partial_{x_i}\phi$ in $\mathscr{D}(\mathbb{R}^n)$, we want to show that their supports are contained in some compact set (which is obvious), and

$$\left| \partial^\alpha \left(\frac{\tau_{-he_i}\phi - \phi}{h} - \frac{\partial}{\partial x_i}\phi \right)(x) \right| = \left| \frac{\tau_{-he_i}\partial^\alpha\phi - \partial^\alpha\phi}{h}(x) - \frac{\partial}{\partial x_i}\partial^\alpha\phi(x) \right| \to 0$$

uniformly as $h \to 0$ for multi-index α. From Mean Value Theorem, it holds that

$$\frac{\tau_{-he_i}\partial^\alpha\phi(x) - \partial^\alpha\phi(x)}{h} = \partial_{x_i}\partial^\alpha(x + \theta h e_i), \quad \theta \in (0,1)$$

and the conclusion follows immediately from the fact that $\partial_{x_i}\partial^\alpha\phi$ is uniformly continuous. $\square$

3.5 Show that for all $f \in \mathscr{D}'(\mathbb{R}^n)$ and $\phi \in \mathscr{D}(\mathbb{R}^n)$ the function $g(x)$ defined as

$$g(y) = \langle f, \tau_{-y}\phi \rangle$$

is in $C^\infty(\mathbb{R}^n)$.

Proof. It suffices to show that $g(y)$ is continuous and $g_{x_i}(y) = \langle f, \tau_{-y}\partial_{x_i}(y) \rangle$.
Since ϕ is uniformly continuous, τ_{-y} is also uniformly continuous and thus $\{\tau_{-(y+h)}\phi - \tau_{-y}\phi\}$ converges to 0 in $\mathscr{D}(\mathbb{R}^n)$. Hence $g(y+h) - g(y) = \langle f, \tau_{-(y+h)}\phi - \tau_{-y}\phi \rangle \to 0$ uniformly, which indicates that g is uniformly continuous.
Now we show that $\{(\tau_{-(y+he_i)}\phi - \tau_{-y}\phi)/h\}$ converges to $\tau_{-y}\partial_{x_i}\phi(x)$ in $\mathscr{D}(\mathbb{R}^n)$ as $h \to 0$. It is obvious that their supports are contained in a common compact set. Also We have from Lagrange's Mean Value Theorem that

$$\frac{\tau_{-(y+he_i)}\phi(x) - \tau_{-y}\phi(x)}{h} = \partial_{x_i}\phi(x + y + \theta h e_i), \quad \theta \in (0,1)$$

Note that $\partial_{x_i}\phi$ is uniformly continuous, we have that

$$\frac{\tau_{-(y+he_i)}\phi(x) - \tau_{-y}\phi(x)}{h} - \tau_{-y}\partial_{x_i}\phi(x) \to 0$$

uniformly as $h \to 0$. Therefore,

$$\begin{aligned}
g_{x_i}(y) &= \lim_{h\to 0} \frac{g(y + he_i) - g(y)}{h} \\
&= \lim_{h\to 0} \frac{\langle f, \tau_{-(y+h)}\phi - \tau_{-y}\phi \rangle}{h} \\
&= \left\langle f, \lim_{h\to 0} \frac{\tau_{-(y+h)}\phi - \tau_{-y}\phi}{h} \right\rangle \\
&= \langle f, \tau_{-y}\partial_{x_i}\phi \rangle.
\end{aligned}$$

$\square$

3.6 Show that for every $f \in \mathscr{S}'$, there exist $u_\alpha \in L^2(\mathbb{R}^n)$ and an even number m such that

$$f = \sum_{|\alpha| \le m} (-1)^{|\alpha|} \tilde{\partial}^\alpha [(1+|x|^2)^{\frac{m}{2}} u_\alpha]$$

Proof. Examining the proof of Lemma 3.2.11 carefully, we can require the m in Lemma 3.2.11 to be even and therefore the m in (3.2.6) and consequently (3.2.7) be even. Therefore, there exists an even m and $u_\alpha \in L^2(\mathbb{R}^n)$ such that

$$\begin{aligned}\langle f, \phi \rangle &= \sum_{|\alpha| \le m} \int_{\mathbb{R}^n} u_\alpha(x) \partial^\alpha \phi(x) (1+|x|^2)^{\frac{m}{2}} dx \\ &= \sum_{|\alpha| \le m} \langle u_\alpha(x)(1+|x|^2)^{\frac{m}{2}}, \partial^\alpha \phi(x) \rangle \\ &= \sum_{|\alpha| \le m} (-1)^{|\alpha|} \tilde{\partial}^\alpha [(1+|x|^2)^{\frac{m}{2}} u_\alpha]\end{aligned}$$ □

4 The Fourier Transform on $\mathscr{S}'$

4.1 Let $H^m(\mathbb{R}) = \{u \in \mathscr{S}' | \tilde{\partial}^\alpha u \in L^2(\mathbb{R}^n)(|\alpha| \le m)\}$, in which the norm is defined as

$$\|u\|_m = \left(\sum_{|\alpha| \le m} \|\tilde{\partial}^\alpha u\|_2^2 \right)^{\frac{1}{2}}.$$

Also we define for each $u \in H^m(\mathbb{R}^n)$

$$\|u\|'_m = \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^m |(\mathscr{F}u)(\xi)|^2 d\xi \right)^{\frac{1}{2}}.$$

Show that

(1) $\|u\|'_m < \infty$;
(2) $\|\cdot\|'_m$ is an equivalent norm in $H^m(\mathbb{R}^n)$;
(3) $H^m(\mathbb{R}^n)$ is complete.

Proof. (1) Since $\tilde{\partial}^\alpha u \in L^2(\mathbb{R}^n)$, we have from Plancherel Theorem that $\mathscr{F}(\tilde{\partial}^\alpha u) \in L^2(\mathbb{R}^n)$, which is $(2\pi i\xi)^\alpha (\mathscr{F}u)(\xi) \in L^2(\mathbb{R}^n)$, which means that $\xi^\alpha (\mathscr{F}u)(\xi) \in L^2(\mathbb{R}^n)$, or, $\int_{\mathbb{R}^n} |\xi|^{2\alpha} |(\mathscr{F}u)(\xi)|^2 d\xi$ exists for all $|\alpha| \le m$. It follows that $\|u\|'_m < \infty$.

(2) Also by Plancherel Theorem it holds that

$$\|\partial^\alpha u\|_2 = \|\mathscr{F}(\tilde{\partial}^\alpha u)\|_2 = \|(2\pi i\xi)^\alpha (\mathscr{F}u)(\xi)\|_2 = 2\pi \|\xi^\alpha (\mathscr{F}u)(\xi)\|_2,$$

thus

$$\|u\|_m = 2\pi \left(\sum_{|\alpha| \le m} \|\xi^\alpha (\mathscr{F}u)(\xi)\|_2^2 \right)^{\frac{1}{2}} = 2\pi \left(\int_{\mathbb{R}^n} \left(\sum_{|\alpha| \le m} |\xi|^{2\alpha} \right) |(\mathscr{F}u)(\xi)|^2 d\xi \right)^{\frac{1}{2}} \le 2\pi \|u\|'_m.$$

On the other hand,

$$\|u\|'_m \le \left(\int_{\mathbb{R}^n} C \left(\sum_{|\alpha| \le m} |\xi|^{2\alpha} \right) |(\mathscr{F}u)(\xi)|^2 d\xi \right)^{\frac{1}{2}} = \sqrt{C} \left(\sum_{|\alpha| \le m} \|\xi^\alpha (\mathscr{F}u)(\xi)\|_2^2 \right)^{\frac{1}{2}} = \frac{\sqrt{C}}{2\pi} \|u\|_m,$$

therefore $\|\cdot\|'_m$ is equivalent to $\|\cdot\|_m$.

(3) Let $\{u_k\}$ be a Cauchy sequence in $H^m(\mathbb{R}^n)$, then $\{\tilde{\partial}^\alpha u_k\}$ is a Cauchy sequence in $L^2(\mathbb{R}^n)$ and thus there exists $u_\alpha \in L^2(\mathbb{R}^n)$ such that $\partial^\alpha u_k \to u_\alpha$ in L^2 norm. Since $L^2(\mathbb{R}^n)$ can be embedded into $\mathscr{S}'$, we have also that $\tilde{\partial}^\alpha u_k \to u_\alpha$ weakly-star. Now we shall show that $\tilde{\partial}^\alpha u_0 = u_\alpha$, which is because

$$\langle \tilde{\partial}^\alpha u_0, \phi\rangle = (-1)^{|\alpha|}\langle u_0, \partial^\alpha \phi\rangle = \lim_{k\to\infty}(-1)^{|\alpha|}\langle u_k, \partial^\alpha\phi\rangle = \lim_{k\to\infty}\langle \tilde{\partial}^\alpha u_k, \phi\rangle = \langle u_\alpha, \phi\rangle$$

for all $\phi \in \mathscr{S}(R^n)$. □

4.2 For any non-negative real s, let

$$H^s(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) | (1+|\xi|^2)^{\frac{s}{2}}\hat{u}(\xi) \in L^2(\mathbb{R}^n),$$

where the norm is defined as

$$\|u\|_s = \|(1+|\xi|^2)^{\frac{s}{2}}\hat{u}(\xi)\|_2.$$

Show that

(1) This definition is equivalent to the original one when $s = m \in \mathbb{N}$;

(2) Inner product $(\cdot,\cdot)$ can be introduced in $H^s(\mathbb{R}^n)$ such that $\|u\|s = (u,u)^{\frac{1}{2}}$;

(3) Let $u \in H^s(\mathbb{R}^n)'$, show that there exists $\tilde{u} \in L^1_{\text{loc}}(\mathbb{R}^n)$ such that

$$\tilde{u}(\xi)(1+|\xi|^2)^{-\frac{s}{2}} \in L^2(\mathbb{R}^n)$$

and

$$\langle u, \mathscr{F}\phi\rangle = \int_{\mathbb{R}^n} \phi(\xi)\cdot\tilde{u}(\xi)d\xi, \quad \forall \phi \in \mathscr{S}.$$

Proof. (1) It follows easily from part (1) and (2) of the previous problem.

(2) Let

$$(u,v) = \int_{\mathbb{R}^n}(1+|\xi|^2)^s\hat{u}(\xi)\overline{\hat{v}(\xi)}d\xi,$$

which is obviously sesqui-linear and conjugate symmetric. The only thing remaining is to show that $(u,u) = 0$ if and only if $u = 0$. `If' is trivial. Now we consider `only if'. Since a nonnegative function with integral zero must be zero almost everywhere, we know that $\hat{u} = 0$ and thus $u = 0$.

(3) First we show that $H^s(\mathbb{R}^n)$ is complete. Suppose u_k is a Cauchy sequence in $H^s(\mathbb{R}^n)$, then $\widehat{u_k}$ is a Cauchy sequence in L^2, so it is also a Cauchy sequence in measure, hence we can find a subsequence $\widehat{u_{k_i}} \to \hat{v}$ almost everywhere. From the proof of the completeness of $L^2(\mathbb{R}^n)$ we know that $\widehat{v} \in L^2$ and $\widehat{u_k} \to \hat{v} \in L^2$, and consequently $u_k \to v$ in L^2 and $v \in L^2$. Similarly, by Fatou's Lemma

$$\int_{\mathbb{R}^n}(1+|\xi|^2)|\widehat{u_k}-\hat{v}|^2 d\xi = \int_{\mathbb{R}^n}\lim_{i\to\infty}(1+|\xi|^2)|\widehat{u_k}-\widehat{u_{k_i}}|^2 d\xi \le \liminf_{i\to\infty}\int_{\mathbb{R}^n}(1+|\xi|^2)|\widehat{u_k}-\widehat{u_{k_i}}|^2 d\xi,$$

whence we see that

$$\lim_{k\to\infty}\int_{\mathbb{R}^n}(1+|\xi|^2)|\widehat{u_k}-\hat{v}|^2 d\xi = 0.$$

It follows that $\|u_k - v\|_s \in H^s(\mathbb{R}^n)$ and thus $v \in H^s(\mathbb{R}^n)$. Hence $H^s(\mathbb{R}^n)$ is a Hilbert space. By Riesz Representation Theorem, there exists $v \in H^s$ such that $\langle u, \phi\rangle = (\phi, v)$ for all $\phi \in H^s \supset \mathscr{S}$.

Now take $\tilde{u}(\xi) = \overline{\hat{v}(-\xi)}(1+|\xi|^2)^s$. Let K be any compact set, suppose that $K \subseteq B(0,R)$ for some R, then

$$\int_K |\tilde{u}| \le (1+R^2)^s\int_K |\hat{v}(-\xi)|d\xi \le (1+R^2)^s\|\hat{v}\|_2 m(K)^{\frac{1}{2}} < \infty,$$

whence we know that $\tilde{u} \in L^1_{\text{loc}}(\mathbb{R}^n)$. And we have from

$$\int_{\mathbb{R}^n} \frac{|\tilde{u}(\xi)|^2}{(1+|\xi|^2)^s} d\xi = \int_{\mathbb{R}^n} (1+|\xi|^2)^s |\hat{v}(\xi)|^2 = \|v\|_2$$

that $\hat{u}(\xi)(1+|\xi|^2)^{-s/2} \in L^2(\mathbb{R}^n)$. Finally,

$$\begin{aligned}
\langle u, \mathscr{F}\phi\rangle = (\mathscr{F}\phi, v) &= \int_{\mathbb{R}^n} (1+|\xi|^2)^s \mathscr{F}(\mathscr{F}\phi)\overline{\mathscr{F}v} d\xi \\
&= \int_{\mathbb{R}^n} (1+|\xi|^2)^s \phi(-\xi) \frac{\tilde{u}(-\xi)}{(1+|\xi|^2)^s} d\xi \\
&= \int_{\mathbb{R}^n} \tilde{u}(\xi)\phi(\xi) d\xi.
\end{aligned}$$

□

4.3 Let $f(x) \in L^1(\mathbb{R}^n)$ show that

$$(\tilde{\mathscr{F}} f)(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi} dx,$$

that is, the Fourier transform of f in $\mathscr{S}'$ is the same as the ordinary Fourier transform.

Proof. Let $\phi \in \mathscr{S}$. Since $\phi \in L^1(\mathbb{R}^n)$, we know that

$$\lim_{R\to\infty} \int_{|x|\le R} \phi(t) e^{-2\pi i t\cdot x} dt = \int_{\mathbb{R}^n} \phi(t) e^{-2\pi i t\cdot x} dt = \mathscr{F}\phi(x)$$

and the convergence is uniform. Since $\mathscr{F}\phi$ is bounded, $\int_{|x|\le R} \phi(t) e^{-2\pi i t\cdot x} dt$ is bounded too if R is large enough. Note that $f \in L^1(\mathbb{R}^n)$, by Lebesgue's Dominated Convergence Theorem we can write

$$\langle \tilde{\mathscr{F}} f, \phi\rangle = \langle f, \mathscr{F}\phi\rangle = \int_{\mathbb{R}^n} f(x) \int_{\mathbb{R}^n} \phi(t) e^{-2\pi i t\cdot x} dt dx = \lim_{R\to\infty} \int_{\mathbb{R}^n} f(x) \int_{|x|\le R} \phi(t) e^{-2\pi i t\cdot x} dt dx$$

Now, since ϕ is bounded, we can apply Fubini's Theorem,

$$\langle \tilde{\mathscr{F}} f, \phi\rangle = \lim_{R\to\infty} \int_{|x|\le R} \int_{\mathbb{R}^n} e^{-2\pi i t\cdot x} f(x) dx \phi(t) dt dx$$

Again by Lebesgue's Dominated Convergence Theorem it holds that

$$\langle \tilde{\mathscr{F}} f, \phi\rangle = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-2\pi i t\cdot x} f(x) dx \phi(t) dt,$$

completing the proof. □

4.4 There is no non-trivial solution to $\Delta f = f$ in $\mathscr{S}'(\mathbb{R}^n)$.

Proof. Take Fourier Transform of both sides, we have $-4\pi^2|\xi|^2 \hat{f}(\xi) = \hat{f}(\xi)$, therefore $\hat{f}(\xi) = 0$ and thus $f = 0$. □

5 Sobolev Spaces

5.1 Verify Theorem 3.5.5 for $\Omega = \mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_n > 0\}$.

5.2 Suppose that $a \in \mathscr{D}$ and $u \in W^{m,p}(\mathbb{R}^n)$, then $a \cdot u \in W^{m,p}(\mathbb{R}^n)$ and there exists a constant C (dependent on a) such that

$$\|a \cdot u\|_{W^{m,p}} \leq C\|u\|_{W^{m,p}}.$$

Proof. By definition

$$\int_{\mathbb{R}^n} \tilde{\partial}^\alpha (au)\phi = (-1)^{|\alpha|} \int_{\mathbb{R}^n} u \cdot a\partial^\alpha \phi$$

Applying integration by parts repeatedly, we see that $\tilde{\partial}^\alpha(au)$ can be written as sum of terms of form $\partial^{\beta_1} a \tilde{\partial}^{\beta_2} u$. Each term is in $L^p(\mathbb{R}^n)$ because $\partial^{\beta_1} a$ is bounded and $\tilde{\partial}^{\beta_2} u$ is in $L^p(\mathbb{R}^n)$, hence the sum is in L^p. The inequality follows easily. ☐

5.3 Suppose that $m \geq l$, show that $W^{m,p}(\Omega) \hookrightarrow W^{l,p}(\Omega)$.

Proof. It is obvious that $W^{m,p}(\Omega) \subseteq W^{l,p}(\Omega)$ and $u \in W^{m,p}(\Omega)$ we have that $\|u\|_{W^{l,p}(\Omega)} \leq \|u\|_{W^{m,p}(\Omega)}$ for all $u \in W^{m,p}(\Omega)$. ☐

5.4 Let $\Omega = (a, b)$ and $f \in L^2(\Omega)$. Prove that there exists a unique $x \in H_0^1(\Omega)$ such that

$$\frac{\tilde{d}^2 x}{dt^2} = f, \tag{3}$$

and $T : f \mapsto x$ is a continuous linear operator from $L^2(\Omega)$ to $H^2(\Omega)$.

Proof. Define

$$J(v) = -\int_a^b v''v - 2\int_a^b fv = \int_a^b |v'|^2 - 2\int_a^b fv, \quad v \in H_0^1(a, b). \tag{4}$$

First we show that if v^* is a minimiser of $J(v)$, then v^* is a solution to (3). Let $\phi \in H_0^1(a, b)$, then

$$J(v^* + \phi) - J(v^*) = 2\int_a^b (v^{*\prime\prime} - f)\phi + \int_a^b \phi''\phi \geq 0$$

hence

$$J(v^* + \epsilon\phi) - J(v^*) = 2\epsilon \int_a^b (v^{*\prime\prime} - f)\phi + \epsilon^2 \int_a^b \phi''\phi \geq 0$$
$$J(v^* - \epsilon\phi) - J(v^*) = -2\epsilon \int_a^b (v^{*\prime\prime} - f)\phi + \epsilon^2 \int_a^b \phi''\phi \geq 0$$

for any $\epsilon > 0$. It must hold that

$$\int_a^b (v^{*\prime\prime} - f)\phi = 0$$

for all $\phi \in H_0^1(a, b)$, and thus $(v^*)'' = f$.

Next we show the existence of the minimiser to (4). Recall that $(u', v')_{L^2}$ is an inner product on $H_0^1(a, b)$. Since $f \in L^2$, $v \mapsto \int_a^b fv$ defines a bounded linear functional, by Riesz representation theorem, there exists $w \in H_0^1(a, b)$ such that

$$\int_a^b fv = (v', w')_{L^2}$$

and thus $J(v)$ can be rewritten as

$$J(v) = \|v - w\|_{H_0^1}^2 - \|w\|_{H_0^1}^2$$

which clearly attains minimum at $v = w$ and nowhere else. Therefore the existence and uniqueness has been proved.

It is clear that T is linear. The boundedness of T follows easily from Poincaré's inequality, which is, in our case, based on the following inequality:

$$\|u\|_{L^\infty} \le C \left\| \frac{\tilde{d}u}{dx} \right\|_{L^2}, \quad \forall u \in H_0^1(a,b) \tag{5}$$

We have seen in Lemma 1.6.15 that (5) holds for all $u \in C_0^\infty(a,b)$. Let $u \in H_0^1(a,b)$. Suppose that $\{u_k\} \subseteq C_0^\infty(a,b)$ converging to u in $H_0^1(a,b)$. It is clear from (5) that $\{u_k\}$ is a Cauchy sequence in $C([a,b])$, and thus $u \in C([a,b])$, and $\|u_k\|_{L^\infty} \to \|u\|_{L^\infty}$. Since $u_k \to u$ in $H_0^1(a,b)$, it naturally holds that $\|\frac{\tilde{d}u_k}{dx}\|_{L^2} \to \|\frac{\tilde{d}u}{dx}\|_{L^2}$. Hence (5) holds for $u \in H_0^1(a,b)$. □

5.5 Let $f(x) \in H_0^1(-1,1)$. Show that

(1) $f(-1) = f(1) = 0$;

(2) $f(x)$ is absolutely continuous;

(3) $f'(x) \in L^2(-1,1)$ (`$'$' means derivative a.e.)

Proof. (1) Given $u \in H_0^1(-1,1)$, there exist functions $u_k \in C_0^\infty(-1,1)$ converging to u in $H_0^1(-1,1)$. Also note that $H^1(-1,1) \hookrightarrow C([-1,1])$ and u_k is mapped to itself, hence u_k converges uniformly to some u^* on $[1,-1]$. Since $u_k(1) = u_k(-1) = 0$ for all k we have that $u^*(1) = u^*(-1) = 0$.

(2) Let $k \to \infty$ in

$$u_k(y) = u_k(x) + \int_x^y u_k'(t)dt$$

and note that

$$\int_x^y |u_k'(t) - u'(t)|dt \le \left(\int_x^y |u_k'(t) - u'(t)|^2 dt \right)^{\frac{1}{2}} (y-x)^{\frac{1}{2}} \le \|u_k - u\|_{H_0^1} (y-x)^{\frac{1}{2}} \to 0$$

we have

$$u(y) = u(x) + \int_x^y u'(t)tdt$$

Hence $u(x)$ is absolutely continuous.

(3) This is obvious, because $u' = (u' - u_k') + u_k$, and $u' - u_k'$ and u_k' are both in $L^2(-1,1)$. □

5.6 Let $f \in H^s(\mathbb{R}^n)$ (See Exercise 4.2 for the definition). Show that if $s > n/2$

(1) $\hat{f}(\xi) \in L^1(\mathbb{R}^n)$;

(2) $f(x)$ equals to a continuous and bounded function on $\mathbb{R}^n$ almost everywhere.

Proof. (1)

$$\int_{\mathbb{R}^n} |\hat{f}(\xi)|d\xi \le \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^s |\hat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^{-s} d\xi \right)^{\frac{1}{2}} \le C \cdot \|f\|_s$$

for some constant C.

(2) For $f \in \mathscr{S}(\mathbb{R}^n)$ it holds that

$$\|f\|_{\mathscr{S}(\mathbb{R}^n)} \leq \int_{\mathbb{R}^n} |\hat{f}(\xi)| d\xi \leq C \cdot \|f\|_s \tag{6}$$

Assume for a moment that $\mathscr{S}(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$. Then for any $f \in H^s(\mathbb{R}^n)$, we can find $\{f_k\} \subset \mathscr{S}(\mathbb{R}^n)$ such that $f_k \to f$ in $H^s(\mathbb{R}^n)$. Equation (6) implies that $\{f_k\}$ is a Cauchy sequence in C^∞-norm, and thus it converges to a bounded continuous function f^* on $\mathbb{R}^n$. It holds that for all $g \in \mathscr{S}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |f - f^*||g| \leq \int_{\mathbb{R}^n} |f - f_k||g| + \int_{\mathbb{R}^n} |f^* - f_k||g| \leq \|f - f_k\|_2 \|g\|_2 + \|f^* - f_k\|_\infty \|g\|_1 \to 0,$$

because, by Plancherel's Theorem, $\|f - f_k\|_2 = \|\hat{f} - \hat{f}_k\|_2 \leq \|f - f_k\|_s$. Hence $f^* = f$ a.e.
Now we show that $\mathscr{S}(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$. Note that $C_0^\infty(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, given $u \in H^s(\mathbb{R}^n)$ there exists $u_k \in C_0^\infty(\mathbb{R}^n)$ such that $\|u_k - u\|_{L^2(\mathbb{R}^n)} \to 0$. Let $v_k = u_k(1 + |\xi|)^{-\frac{s}{2}}$, then $v_k \in C_0^\infty(\mathbb{R}^n)$ and $v_k(1+|\xi|)^{\frac{s}{2}} \to u$ in $L^2(\mathbb{R}^n)$. Since $v_k \in \mathscr{S}(\mathbb{R}^n)$, there exist $w_k \in \mathscr{S}(\mathbb{R}^n)$ such that $v_k = \hat{w_k}$ (actually w_k is the inverse Fourier transform of v_k). Hence $\hat{w_k}(1 + |\xi|)^{\frac{s}{2}} \to u$ in $L^2(\mathbb{R}^n)$, that is, $w_k \to u$ in $H^s(\mathbb{R}^n)$. □

5.7 Let $m \in \mathbb{N}$, define

$$H^{-m} = \{f \in \mathscr{S}' : (1 + |\xi|^2)^{-\frac{m}{2}} \hat{f}(\xi) \in L^2(\mathbb{R}^n)\},$$

and the norm

$$\|f\|_{-m} = \|(1 + |\xi|^2)^{-\frac{m}{2}} \hat{f}(\xi)\|_{L^2(\mathbb{R}^n)}.$$

Show that any $f \in H^{-m}$ can be written as the sum of the derivatives of finitely many functions in $L^2(\mathbb{R}^n)$.

Proof. It suffices to show that $H^{-m}(\mathbb{R}^n)$ defined in this way is equivalent to $H^m(\mathbb{R}^n)'$, then the conclusion follows from Corollary 3.5.13 because $H^m(\mathbb{R}^n) = H_0^m(\mathbb{R}^n)$.
By Riesz Representation Theorem, for $v \in (H^m)'$ there exists $u_v \in H^m$ such that $v[u] = (u, u_v)_{H^m}$ for all $u \in H^m$. Note that $\mathscr{F}^{-1}((1+|\xi|^2)^m \widehat{u_v}) \in H^{-m}$, hence there exists a natural bijection between $(H^m)'$ and H^{-m}. Finally, since

$$\|\mathscr{F}^{-1}((1 + |\xi|^2)^m \widehat{u_v})\|_{H^{-m}} = \|(1 + |\xi|^2)^{\frac{m}{2}} \widehat{u_v}\|_{L^2} = \|u_v\|_{H^m}$$

it follows that H^{-m} and $(H^m)'$ are, in fact, isomorphic. □