# 1 Definition and Basic Properties of Compact Operator

1.1 Let $\mathscr{X}$ be a infinite dimensional Banach space. Show that if $A \in \mathfrak{C}(\mathscr{X})$, $A$ does not have bounded inverse.

*Proof.* Denote the unit ball of $\mathscr{X}$ by $B$ and the unit sphere $S$. Suppose that $\{x_n\} \subseteq S$, then $\|x_n\| \leq \|A^{-1}\| \|Ax_n\|$ for all $n$. Because $\{Ax_n\}$ has a convergent subsequence, we know that $x_n$ has a convergent subsequence too, which implies that $S$ is sequentially compact. This contradicts with Theorem 1.4.28, which states that a normed linear space is finite dimensional iff the unit sphere is sequentially compact. □

1.2 Let $\mathscr{X}$ be a Banach space and $A \in \mathscr{L}(\mathscr{X})$ satisfy $\|Ax\| \geq a\|x\|$ for all $x \in \mathscr{X}$, where $a$ is a positive constant. Prove that $A \in \mathfrak{C}(\mathscr{X})$ iff $\mathscr{X}$ is finite-dimensional.

*Proof.* It suffices to show that $A \in \mathfrak{C}(\mathscr{X})$ iff every bounded set in $\mathscr{X}$ is sequentially compact.

`If': Let $\{x_n\}$ be a bounded sequence, thus $Ax_n$ is bounded since $A$ is bounded, thus it has a convergent subsequence and $A$ is therefore compact.

`Only if': Let $B$ be a bounded set and $\{x_n\} \subseteq B$. We can find a convergent subsequence in $\{Ax_n\}$, say $Ax_{n_k}$. Note that $\|Ax_n\| \geq a\|x_n\|$, we know that $\|x_{n_k}\|$ is a Cauchy sequence thus convergent (as $\mathscr{X}$ is complete). □

1.3 Let $\mathscr{X}$ and $\mathscr{Y}$ be Banach spaces, $A \in \mathscr{L}(\mathscr{X}, \mathscr{Y})$, $K \in \mathfrak{C}(\mathscr{X}, \mathscr{Y})$ and $R(A) \subseteq R(K)$. Show that $A \in \mathfrak{C}(\mathscr{X}, \mathscr{Y})$.

*Proof.* Let $K' : \mathscr{X} / \ker K \to \mathscr{X}$ be the canonical map, then $K'$ is also a compact operator, since $B + \ker K$ is the unit ball in $\mathscr{X} / \ker K$, where $B$ is the unit ball in $\mathscr{X}$. Note that $K'$ is continuous, thus $K'^{-1}$ is a closed map, and $D(K'^{-1}) = R(K) \supseteq R(A)$, hence $K'^{-1}A : \mathscr{X} \to \mathscr{X} / \ker K$ is a closed map, and its domain is the entire $\mathscr{X}$, thus from the closed graph theorem that $K'^{-1}A$ is continuous, whence it follows that $A = K(K'^{-1}A)$ is compact. □

1.4 Let $H$ be a Hilbert space and $A : H \to H$ is a compact operator. Suppose that $x_n \rightharpoonup x_0$ and $y_n \rightharpoonup y_0$. Show that $(x_n, Ay_n) \to (x_0, Ay_0)$.

*Proof.* We have that $|(x_n, Ay_n) - (x_0, Ay_0)| \leq |(x_n, Ay_n - Ay_0)| + |(x_n - x_0, Ay_0)|$. Since $x_n \rightharpoonup x_0$, it is clear that $\{x_n\}$ is bounded, say by $M$, and the second term goes to 0. Since $y_n \rightharpoonup y_0$ and $A$ is compact (thus completely continuous), we have that $Ay_n \to Ay_0$. Notice that $|(x_n, Ay_n - Ay_0)| \leq \|x_n\| \|Ay_n - Ay_0\| \leq M\|Ay_n - Ay_0\|$, thus the first term also goes to 0. □

1.5 Let $\mathscr{X}, \mathscr{Y}$ be Banach spaces and $A \in \mathscr{L}(\mathscr{X}, \mathscr{Y})$. Suppose that $R(A)$ is closed and infinite-dimensional. Show that $A \notin \mathfrak{C}(\mathscr{X}, \mathscr{Y})$.

*Proof.* Suppose that $A$ is compact. Note that $R(A)$ is a Banach space, and there exist a bounded set $B$ which is not sequentially compact, since $R(A)$ is infinite dimensional. Take $\{y_n\} \subseteq B$ such that it has no convergent subsequence. Consider $\mathscr{X} / \ker A$ and $A' : \mathscr{X} / \ker A \to R(A)$ is the induced natural map, which is bijective. Let $[x_n] = A'^{-1}(y_n)$, we can choose $x_n \in [x_n]$ such that $Ax_n = y_n$ and $\|x_n\| \leq 2\|[x_n\| \leq 2\|A'^{-1}\| \|y_n\|$, thus $\{x_n\}$ is bounded. We meet a contradiction. Therefore $A$ can not be compact. □

1.6 Let $w_n \in \mathbb{K}$ with $w_n \to 0$. Show that the map defined as

$$T : \{\xi_n\} \mapsto \{w_n \xi_n\}$$

is a compact operator on $l^p (p \geq 1)$.

*Proof.* It is clear that $T \in \mathscr{L}(l^2)$ since $\{w_n\}$ is bounded. Let $T_n$ be a linear operator defined on $l^p$ as

$$T_n : (\xi_1, \dots, \xi_n, \xi_{n+1}, \dots) \mapsto (w_1\xi_1, \dots, w_n\xi_n, 0, 0, \dots).$$

Since $\dim T_n(l^2) < \infty$, $T_n$ has finite-rank. It is also bounded, thus compact. Given $\epsilon > 0$, there exists $N$ such that $|w_n| < \epsilon$ for all $n > N$. It then follows that $\|T_n x - Tx\| \leq \epsilon\|x\|$, thus $\|T_n - T\| \leq \epsilon$, and $\|T_n - T\| \to 0$. Because $\mathfrak{C}(l^2)$ is closed, $T$ is compact. □

1.7 Let $\Omega \subset \mathbb{R}^n$ be a measurable set and $f$ be a bounded measurable function on $\Omega$. Prove that $F : x(t) \mapsto f(t)x(t)$ is a compact operator on $L^2(\Omega)$ iff $f = 0$ almost everywhere on $\Omega$.

*Proof.* 'If': Trivial.

'Only if': Assume that $mQ > 0$. If $f(x) > 0$ on a set $A$ with $mA > 0$, we can find a compact set $C \subset A$ with $mC > 0$. Then $f$ is bounded below on $C$, say, $f(x) \geq c > 0$ for all $x \in C$. We can find $\{x_n\} \subseteq L^2(C)$ such that $\|x_n\|_2 = 1$ while $x_n \rightharpoonup 0$ (for instance, take an orthonormal basis). Since $F$ is completely continuous, we have

$$\|Fx\|^2 = \int_\Omega |f(t)|^2 |x_n(t)|^2 \to 0.$$

On the other hand,

$$\|Fx\|^2 \geq \int_C |f(t)|^2 |x_n(t)|^2 \geq c^2 \int_C |x_n(t)|^2 = c^2,$$

contradiction. □

1.8 Let $\Omega \subset \mathbb{R}^n$ be a measurable set and $K \in L^2(\Omega \times \Omega)$. Show that

$$A : u(x) \mapsto \int_\Omega K(x, y)u(y)dy, \quad \forall u \in L^2(\Omega)$$

is a compact operator on $L^2(\Omega)$.

*Proof.* It is clear that $L^2(\Omega)$ is separable, hence there exists an orthonormal basis $\{u_i\} \subset L^2(\Omega)$. Then

$$K(x, y) = \sum_{i=1}^{\infty} K_i(y)u_i(x),$$

where

$$K_i(y) = \int_\Omega K(x, y)u_i(x).$$

for almost all $y$. The Parseval identity gives that

$$\int_\Omega |K(x, y)|^2 dx = \sum_{i=1}^{\infty} |K_i(y)|^2$$

and thus

$$\int_{\Omega\times\Omega} |K(x, y)|^2 dxdy = \sum_{i=1}^{\infty} \int_\Omega |K_i(y)|^2 dy. \tag{1}$$

We now define the following operator of rank $N$

$$A_N u = \int_\Omega K_N(x, y)f(y)dy,$$

where

$$K_N(x,y) = \sum_{i=1}^{N} K_i(y)u_i(x).$$

By Cauchy-Schwarz inequality,

$$\begin{aligned}
\|(A - A_N)\|^2 &\le \int_{\Omega\times\Omega} |K(x,y) - K_N(x,y)|^2 dxdy \\
&= \int_{\Omega\times\Omega} |K(x,y)|^2 dxdy - 2\int_{\Omega\times\Omega} K(x,y)\sum_{i=1}^{N} K_i(y)u_i(x)dxdy + \sum_{i=1}^{N}\int_{\Omega} |K_i(y)|^2 dy \\
&= \int_{\Omega\times\Omega} |K(x,y)|^2 dxdy - \int_{\Omega} |K_i(y)|^2 dy \to 0
\end{aligned}$$

as $N \to \infty$. Hence $A_N \to A$ and $A$ is therefore compact. □

1.9 Let $H$ be a Hilbert space, $A \in \mathfrak{C}(H)$, $\{e_n\}$ is an orthonormal set in $H$. Show that $\lim_{n\to\infty}(Ae_n, e_n) = 0$.

*Proof.* It can be proved that $e_n \rightharpoonup 0$ (See proof to Exercise 2.5.19), thus the conclusion follows from Exercise 4.1.4. □

1.10 Let $\mathscr{X}$ be a Banach space, $A \in \mathfrak{C}(H)$, $\mathscr{X}_0$ is a closed subspace of $\mathscr{X}$ such that $A(\mathscr{X}_0) \subseteq \mathscr{X}_0$. Prove that the map $T : [x] \mapsto [Ax]$ is a compact operator on $\mathscr{X}/\mathscr{X}_0$.

*Proof.* It can be proved that $B + \ker A$ is the unit ball in $\mathscr{X}/\ker A$, where $B$ is the unit ball in $\mathscr{X}$. Let $\{[x_n]\}$ be a bounded sequence, we can find $\{x_n\}$ such that $\|x_n\| \le 2\|[x_n]\|$, thus $\{x_n\}$ is bounded, and $\{Ax_n\}$ has a convergent subsequence, thus $\{T[x_n]\} = \{[Ax_n]\}$ has also a convergent subsequence. $T$ is compact. □

1.11 Let $\mathscr{X}, \mathscr{Y}, \mathscr{Z}$ be Banach spaces, $\mathscr{X} \subseteq \mathscr{Y} \subseteq \mathscr{Z}$, if the embedding map from $\mathscr{X}$ to $\mathscr{Y}$ is compact and from $\mathscr{Y}$ to $\mathscr{Z}$ continuous. Prove that for any $\epsilon > 0$, there exists $c(\epsilon) > 0$ such that

$$\|x\|_{\mathscr{Y}} \le \epsilon\|x\|_{\mathscr{X}} + c(\epsilon)\|x\|_{\mathscr{Z}}, \quad \forall x \in \mathscr{X}.$$

*Proof.* Prove by contradiction. Suppose that there exists $\epsilon_0$, for all $n$ there exists $x_n \in \mathscr{X}$ such that $\|x_n\|_{\mathscr{Y}} > \epsilon_0\|x_n\|_{\mathscr{X}} + n\|x_n\|_{\mathscr{Z}}$. Let $y_n = x_n/\|x_n\|$, then it holds that $\|y_n\|_{\mathscr{Y}} > \epsilon_0 + n\|y_n\|_{\mathscr{Z}}$. Since the embedding map $\mathscr{X} \to \mathscr{Y}$ is compact and $\|y_n\| = 1$ for all $n$, we know that $\|y_n\|_{\mathscr{Y}}$ is bounded thus $\|y_n\|_{\mathscr{Z}} \to 0$. Also we know that $\{y_n\}$ has a convergent subsequence in $\mathscr{Y}$, say $y_{n_k} \to y$ in $\mathscr{Y}$ as $k \to \infty$. Then $z_{n_k} \to y$ in $\mathscr{Z}$ as the embedding map from $\mathscr{Y}$ to $\mathscr{Z}$ is continuous, and therefore $y$ must be 0. But $\|y_{n_k}\| \ge \epsilon$, we reach a contradiction. □

# 2 Riesz-Fredholm Theory

2.1 Let $\mathscr{X}$ be a Banach space and $M \subseteq \mathscr{X}$ is a closed linear subspace with $\operatorname{codim} M = n$. Show that there exists linearly independent set $\{\phi_k\}_{k=1}^{n} \subseteq \mathscr{X}^*$ such that

$$M = \bigcap_{k=1}^{n} N(\phi_k).$$

*Proof.* Let $\{e_i + M\}$ $(i = 1, \dots, n)$ be a basis of $\mathscr{X}/M$, $D = \{e_1, \dots, e_n\}$, $D_i = \overline{\operatorname{span}\{D \setminus \{e_i\}, M\}}$. Then we have that $e_i \notin D_i$ and we can a bounded linear functional $\phi_i$ such that $\phi_i(D_i) = 0$ and $\phi_i(e_i) = 1$. It is easy to verify that $\{\phi_i\}$ is linearly independent, and $M = \bigcap_{i=1}^{n} N(\phi_i)$. □

2.2 Let $\mathscr{X}, \mathscr{Y}$ be Banach space and $T \in \mathscr{L}(\mathscr{X}, \mathscr{Y})$ is surjective. Define $\widetilde{T} : \mathscr{X}/N(T) \to \mathscr{Y}$ as

$$\widetilde{T}[x] = Tx, \quad \forall x \in [x], \forall [x] \in \mathscr{X}/N(T);$$

Show that $\widetilde{T}$ is a linear homeomorphism.

*Proof.* It is clear that $\widetilde{T}$ is well-defined and linear. For each $[x]$ we can find $x \in [x]$ such that $\|x\| \leq 2\|[x]\|$. Thus $\|\widetilde{T}[x]\| = \|Tx\| \leq \|T\|\,\|x\| \leq 2\|T\|\,\|[x]\|$ and $\widetilde{T}$ is continuous. Also $\widetilde{T}$ is a bijection, hence it is a homeomorphism. □

2.3 Let $\mathscr{X}$ be a Banach space, $M, N_1, N_2$ be closed linear subspaces of $\mathscr{X}$. Suppose that

$$M \oplus N_1 = \mathscr{X} = M \oplus M_2,$$

show that $N_1$ is homeomorphic to $N_2$.

*Proof.* It suffices to show that $N_1$ and $N_2$ are both homeomorphic to $\mathscr{X}/M$. Define $F : N_1 \to \mathscr{X}/M$ as $F(x) = x + M$, and it is easy to verify that $F$ is well-defined. Since $\mathscr{X} = M \oplus N_1$, $F$ is bijective. Besides, it holds that $\|F(x)\| = \|x + M\| = \inf_{m \in M} \|x + m\| \leq \|x\|$, whence we know that $F$ is continuous thus a homeomorphism. □

2.4 Let $A \in \mathfrak{C}(\mathscr{X})$, $T = I - A$, show that

(1) $\forall x \in \mathscr{X}/N(T)$, $\exists x_0 \in [x]$, such that $\|x_0\| = \|[x]\|$;

(2) Suppose that $y \in \mathscr{X}$ such that $Tx = y$ has at least one solution, show that one of the solutions has the minimum norm.

*Proof.* (1) Since $\|[x]\| = \inf_{z \in N(T)} \|x + z\|$, we can choose $z_n \in N(T)$ for each $n$ such that $\|x + z_n\| < \|[x]\| + \frac{1}{n}$, so $\{x + z_n\}$ is bounded. Since $A$ is compact, we have that $\{Ax + Az_n\} = \{Ax + z_n\}$ has a convergent subsequence, say $Ax + z_{n_k} \to z$, thus $z_{n_k} \to z - Ax$. It follows that $\|x + z_{n_k}\| \to \|Tx + z\|$, combining with $\|x + z_n\| \to \|[x]\|$ we have that $\|Tx + z\| = \|[x]\|$. We verify that $Tx + z \in [x]$, or, $Tx + z - x \in N(T)$: $T(Tx + z - x) = T(z - Ax) = \lim T(z_{n_k}) = 0$.

(2) Suppose that $x'$ is a solution to $Tx = y$, then the set of all the solutions is exactly $[x']$. From (1) we know that there exists $x_0 \in [x']$, thus a solution to $Tx = y$, with the minimum norm $\|[x']\|$. □

2.5 Let $A \in \mathfrak{C}(\mathscr{X})$ and $T = I - A$. Show that

(1) $N(T^k)$ is finite dimensional; and

(2) $R(T^k)$ is closed

for all $k \in \mathbb{N}$.

*Proof.* $T^k = (I - A)^k = I - A_k$, where $A_k$ is compact, as a result of Proposition 4.1.2(2) and (6). □

2.6 Let $M$ be a closed linear subspace on Banach space $B$. Call a bounded linear operator $P : \mathscr{X} \to M$ with $P^2 = P$ a projection operator on $M$. Show that

(1) If $M$ is finite dimensional then a projection operator on $M$ do exist;

(2) If $P$ is a projection operator on $M$ then $I - P$ is a projection operator on $R(I - P)$ from $\mathscr{X}$;

(3) If $P$ is a projection operator on $M$ then $\mathscr{X} = M \oplus N$, where $N = R(I - P)$;

(4) If $A \in \mathfrak{C}(\mathscr{X})$ and $T = I - A$, then it holds that

$$N(T) \oplus \mathscr{X}/N(T) = \mathscr{X} = R(T) \oplus \mathscr{X}/R(T)$$

in the sense of isomorphism both algebraical and topological.

*Proof.* (1) Let $e_1, \dots, e_n$ be a normal basis of $M$. From Hahn-Banach Theorem, there exists continuous linear functionals $f_1, \dots, f_n$ such that $f_k(e_j) = \delta_{kj}$. Then define $Px = \sum f_k(x)e_k$, and it is easy to verify that $P$ is bounded and satisfies that $P^2 = P$.

(2) The conclusion follows from that $(I - P)^2 = I - 2P + P^2 = I - 2P + P = I - P$.

(3) It is clear that $\mathscr{X} = M + R(I-P)$ and we shall show that $M \cap R(I-P) = 0$. Let $x \in M \cap R(I-P)$, then there exists $y$ such that $x = (I-P)y$, thus $Px = P(I-P)y = (P - P^2)y = 0$. Since $x \in M$, it holds that $x = Px$, whence we obtain that $x = 0$.

(4) Since $N(T)$ is finite dimensional, from (1) there exists a projection operator $P$ on $N(T)$, and from (3) it suffices to show that $\mathscr{X}/N(T)$ is isomorphic to $R(I - P)$.
Let $F : \mathscr{X}/N(T) \to R(I - P)$ be defined as $F([x]) = (I - P)x$. It is clear that $F$ is well-defined, bijective and linear (algebraically isomorphic). For all $[x] \in \mathscr{X}/N(T)$ there exists $x' \in [x]$ such that $\|x'\| \leq 2\|[x]\|$, so $\|F([x])\| = \|(I - P)x'\| \leq \|I - P\| \, \|x'\| \leq 2\|I - P\| \, \|[x]\|$, and thus $F$ is continuous (topologically homeomorphic).
Therefore we obtain that $N(T) \oplus \mathscr{X}/N(T) = \mathscr{X}$.
Since codim $R(T) = \dim N(T)$, we know that $\mathscr{X}/R(T)$ and $N(T)$ are isomorphic both algebraically and topologically. And it is obvious that $R(T)$ is isomorphic to $\mathscr{X}/N(T)$, since the map $y = Tx \mapsto [x]$ is an isomorphism. Thus we also have that $\mathscr{X}/R(T) \oplus R(T) = \mathscr{X}$. $\square$

# 3 Spectrum Theory of Compact Operators (Riesz-Schauder Theory)

($\mathscr{X}$ denotes Banach space in this section)

3.1 Given sequence of numbers $\{a_n\}$ and define operator $A$ on $l^2$ as

$$A : (x_1, x_2, \dots) \mapsto (a_1 x_1, a_2 x_2, \dots).$$

(1) Show that $A \in \mathscr{L}(l^2)$ iff $\{a_n\}$ is bounded;

(2) If $A \in \mathscr{L}(l^2)$ find $\sigma(A)$ and the types of the spectral points.

*Proof.* (1) `If': Suppose that $|a_n| \leq M$, then $\|Ax\| \leq M\|x\|$.
`Only if': If $\{a_n\}$ is not bounded, then there exists $n_1 < n_2 < \cdots$ such that $|a_{n_k}| > k$. For each $m$, Take $x_m = (\xi_1, \xi_2, \dots)$ where $\xi_{n_m} = 1$ and $\xi_j = 0$ for all of the rest indices $j$. It is clear that $x \in l^2$ and $\|x_m\| = 1$. We compute $\|Ax_m\| > m^{\frac{1}{2}} \to \infty$ as $m \to \infty$, which contradicts with the continuity of $A$. Therefore $\{a_n\}$ is bounded.

(2) Since $A \in \ell^2$, we know that $\{a_n\}$ is bounded, say, by $M$.
If $\lambda = a_i$ for some $i$, then $(\lambda I - A)x = 0$ has nonzero solutions and $\lambda \in \sigma_p(A)$. Now assume that $\lambda \neq a_i$ for all $i$, then $(\lambda I - A)^{-1}$ exists, sending $(x_1, x_2, \dots)$ to $(\frac{x_1}{\lambda - a_1}, \frac{x_2}{\lambda - a_2}, \dots)$.
If $\lambda$ is not a limit point of $\{a_i\}$, then $\frac{1}{|\lambda - a_i|}$ is bounded away from 0, so $(\frac{x_1}{\lambda - a_1}, \frac{x_2}{\lambda - a_2}, \dots) \in \ell^2$ whenever $(x_1, x_2, \dots) \in L^2$ and $R(\lambda I - A) = \ell^2$, thus $\lambda \notin \sigma(A)$.
Now let $\lambda \neq a_i$ be a limit point of $\{a_i\}$, suppose that $|a_{n_k} - \lambda| < \frac{1}{k}$, where $\{a_{n_k}\}$ are pairwise distinct. Consider $x = (x_1, x_2, \dots)$ with $x_{n_k} = (\lambda - a_{n_k})/\sqrt{k}$ and $x_i = 0$ for $i \neq n_k$, then $\sum x_i^2 = \sum (\lambda - a_{n_k})^2/k < \sum 1/k^3 < \infty$. However, $\sum x_i^2/(\lambda - a_i)^2 = \sum 1/k = \infty$, hence $x \notin R(\lambda I - A)$. Note that any $x$ with finitely many non-zero components is in $R(\lambda I - A)$, we know that $R(\lambda I - A)$ is dense in $\ell^2$. Therefore, $\lambda \in \sigma_c(A)$.
We conclude that $\sigma(A) = \overline{\{a_i\}}$ with $\sigma_p(A) = \{a_i\}$ and $\sigma_c(A)$ the rest spectral points. $\square$

3.2 In $C[0,1]$ consider the operator

$$T : x(t) \mapsto \int_0^t x(s)ds, \quad \forall x(t) \in C[0,1].$$

(1) Show that $T$ is a compact operator;

(2) Find $\sigma(T)$ and a nontrivial closed invariant subspace of $T$.

*Proof.* (1) It suffices to show that $T(B_1)$ is sequentially compact, or, uniformly bounded and equi-continuous. First we have that $\|Tx\| \le \left|\int_0^1 x(s)ds\right| \le \|x\|$ implying that $T(B_1)$ is uniformly bounded. Besides it holds that $|(Tx)(s') - (Tx)(s'')| = \left|\int_{s'}^{s''} x(s)ds\right| \le \|x\| \, |s'' - s'|$ implying that $T(B_1)$ is equi-continuous.

(2) First of all, $\|T\| = 1$ implies that $\sigma(T)$ is contained in the closed disc. Since $C[0,1]$ is infinite-dimensional, we know that $0 \in \sigma(T)$. Any other spectral point must be eigenvalue, that is, if $\lambda \neq 0$ belongs to $\sigma(T)$, then $Tx = \lambda x$ has non-zero solution. But $Tx = \lambda x$ has only zero solution, hence $\sigma(T) = \{0\}$. An invariant space of $T$ is $C^1[0,1]$. □

3.3 Let $A \in \mathfrak{C}(\mathscr{X})$. Prove that $x - Ax = 0$ has only zero solution iff $x - Ax = y$ has solution for all $y \in \mathscr{X}$.

*Proof.* `Only if': This is Theorem 4.2.6.
`If': Let $T = I - A$, then $\dim N(T) = \operatorname{codim} R(T) = 0$, thus $N(T) = \{0\}$. □

3.4 Let $T \in \mathscr{L}(\mathscr{X})$ and there exists $m \in \mathbb{N}$ such that

$$\mathscr{X} = N(T^m) \oplus R(T^m).$$

Show that $p(T) = q(T) \le m$.

*Proof.* Let $x \in N(T^{m+1})$. We have that $T^m x \in R(T^m) \cap N(T^m)$, yielding that $T^m x = 0$ and $x \in N(T^m)$. So $N(T^{m+1}) \subseteq N(T^m)$, thus $p(T) \le m$.

Now we show that $q(T) = p(T)$. First we show that $q(T) \ge p(T)$. For simplicity, we use notations $p$ and $q$ instead of $p(T)$ and $q(T)$ respectively.

(1) Proof of $p \le q$. We have that $T(R(T^q)) = R(T^{q+1}) = R(T^q)$, thus for $y \in R(T^q)$, we have $x \in R(T^q)$ such that $Tx = y$.
We claim that if $Tx = 0$ for some $x \in R(T^q)$ then $x$ must be zero. If not, there exists $x_1 \in R(T^q) \backslash \{0\}$ such that $Tx_1 = 0$, then there exists $x_2 \neq 0$ such that $Tx_2 = x_1$. Continuing this process, we obtain $\{x_n\}$ such that $0 \neq x_1 = Tx_2 = \cdots = T^{n-1}x_n$, but $0 = Tx_1 = T^n x_n$. Thus $x_n \notin N(T^{n-1})$ and $x_n \in N(T^n)$ for all $n$, which contradicts with $p < \infty$.
Now we show that $N(T^{q+1}) = N(T^q)$, which would imply that $p \le q$. It suffices to show that $N(T^{q+1}) \subseteq N(T^q)$. Let $x \in N(T^{q+1})$. Since $T^q x \in R(T^q)$ and $T(T^q x) = 0$, we must have that $T^q x = 0$ and $x \in N(T^q)$.

(2) Proof of $p \ge q$. This is obviously true for $q = 0$. Assume that $q > 0$. It suffices to show that $N(T^{q-1}) \subsetneq N(T^q)$. Let $y \in R(T^{q-1}) \setminus R(T^q)$. Then there exists $x$ such that $y = T^{q-1}x$, and there also exists $z$ such that $Ty = T^{q+1}z$ since $Ty \in R(T^q) = R(T^{q+1})$. Thus $T^{q-1}(x - Tz) = y - T^q z \neq 0$ because $y \notin R(T^q)$. So $x - Tz$ does not belong to $N(T^{q-1})$. And it is obvious that it belongs to $N(T^q)$, which establishes that $N(T^{q-1}) \subsetneq N(T^q)$. □

3.5 Let $A, B \in \mathscr{L}(\mathscr{X})$ and $AB = BA$. Prove that

(1) $R(A)$ and $N(A)$ are invariant subspaces of $B$;

(2) $R(B^n)$ and $N(B^n)$ are invariant subspaces of $B$ for all $n \in \mathbb{N}$.

*Proof.* (1) Let $y \in R(A)$, then $y = Ax$ for some $x$. It holds that $By = BAx = A(Bx) \in R(A)$, thus $R(A)$ is an invariant subspace of $B$.
Let $y \in N(A)$, then $A(By) = B(Ay) = 0$, indicating that $By \in N(A)$. Hence $N(A)$ is an invariant subspace of $B$.

(2) The conclusion follows from $B(B^n x) = B^n(Bx) \in R(B^n)$ and $B^n(By) = B(B^n y) = 0$ for $y \in N(B^n)$. □

3.6 Let $A \in \mathscr{L}(\mathscr{X})$ and $M$ is a finite-dimensional invariant subspace of $A$. Show that

(1) The action of $A$ on $M$ can be described by a matrix;
(2) At least one eigenvector of $A$ is in $M$.

*Proof.* Trivial, as $A|_M$ can be viewed as a linear transformation over $M$ (which is finite dimensional). □

3.7 Let $x_0 \in \mathscr{X}$ and $f \in \mathscr{X}^*$ satisfy $\langle f, x_0 \rangle = 1$. Let $A = x_0 \otimes f$ and $T = I - A$. Find $p(T)$.

*Proof.* We have that $Ax = \langle f, x \rangle x_0$ and $A^2 = A$. Thus $N(T) = N(T^2)$. If $\dim \mathscr{X} > 1$ then $N(T) \neq \mathscr{X}$ so $p(T) = 1$; otherwise $N(T) = \mathscr{X}$, so $p(T) = 0$. □

# 4 Hilbert-Schmidt Theorem

($H$ denotes complex Hilbert space in this section)

4.1 Let $A \in \mathscr{L}(H)$, show that $A + A^*$, $AA^*$ and $A^*A$ are all symmetric and $\|AA^*\| = \|A^*A\| = \|A\|^2$.

*Proof.* It is trivial to prove that $A + A^*$, $AA^*$ and $A^*A$ are symmetric. With respect to norm, we have $\|AA^*\| = \sup_{\|x\|=1} |(AA^*x, x)| = \sup_{\|x\|=1} |(A^*x, A^*x)\| = \sup_{\|x\|=1} \|A^*x\|^2 = \|A^*\|^2 = \|A\|^2$. Similarly we have $\|A^*A\| = \|A\|^2$. □

4.2 Let $A \in \mathscr{L}(H)$ satisfying $(Ax, x) \geq 0$ for all $x \in H$ and $(Ax, x) = 0$ iff $x = 0$. Show that

$$\|Ax\|^2 \leq \|A\|(Ax, x), \quad \forall x \in H.$$

*Proof.* It is not hard to show that the following generalized Cauchy's Inequality holds.

$$|(Au, v)|^2 \leq (Au, u)(Av, v).$$

Let $u = x$ and $v = Ax$, we have $|(Au, Au)|^2 \leq (Ax, x)(A^2x, Ax) \leq (Ax, x) \cdot \|A\| \cdot \|Ax\|^2$, which simplifies to our desired result. □

4.3 Let $A$ be a symmetric compact operator on $H$, and

$$m(A) = \inf_{\|x\|=1} (Ax, x), \quad M(A) = \sup_{\|x\|=1} (Ax, x)$$

Prove that

(1) If $m(A) \neq 0$ then $m(A) \in \sigma_p(A)$;
(2) If $M(A) \neq 0$ then $M(A) \in \sigma_p(A)$;

*Proof.* Consider $A_\alpha = A + \alpha I$, then the spectrum is translated by $\alpha$, so $m(A_\alpha) = m(A) + \alpha$ and $M(A_\alpha) = M(A) + \alpha$. For $\alpha < 0$ small enough, $m(A_\alpha) < M(A_\alpha) < 0$. Suppose that $(A_\alpha x_n, x_n) \to m(A_\alpha)$ with $\|x_n\| = 1$. From Proposition 4.4.5(5), it holds that $\|A_\alpha\| = -m(A_\alpha)$. Note that

$$\|A_\alpha x_n - m(A_\alpha)x_n\|^2 = \|A_\alpha\|^2 - 2m(A_\alpha)(A_\alpha x_n, x_n) + m(A_\alpha)^2 \to 0$$

as $n \to \infty$, it follows that $m(A_\alpha)$ is in the spectrum of $A_\alpha$. Hence $m(A)$ is in $\sigma(A)$, and $A$ is compact, thus if $m(A) \neq 0$ it must be in $\sigma_p(A)$.

Similarly consider $A + \alpha I$ for $\alpha > 0$ enough, it yields that $M(A) \in \sigma_p(A)$ if $M(A) \neq 0$. □

4.4 Let $A$ be a symmetric compact operator, show that

(1) If $A$ is non-zero then it has at least one non-zero eigenvalue;

(2) If $M$ is an non-trivial invariant subspace then $M$ contains some eigenvector of $A$.

*Proof.* (1) It follows directly from Theorem 4.4.6.

(2) Assume $M$ is closed, then $A|_M$ is compact and symmetric. Since $M$ is nontrivial, $A|_M$ is non-zero, and therefore has an eigenvalue on $M$.

□

4.5 Show that $P \in \mathscr{L}(H)$ is an orthogonal projector if and only if

(1) $P$ is symmetric, i.e., $P = P^*$;

(2) $P$ is idempotent, i.e., $P^2 = P$.

*Proof.* `Only if': Trivial.

`If': Let $M = \{x : Px = x\}$, then $M$ is a linear subspace of $H$. Since $P$ is continuous, it follows that $M$ is closed. If $Px = y$ then $Py = P^2x = Px = y$, which means that $M$ is the range of $P$. Now notice that $(Py, x - Px) = (y, P^*x - P^*Px) = (y, Px - P^2x) = 0$, so $R(P) \perp R(I - P)$. Also $x = Px + (x - Px)$, we see that $Px$ is an orthogonal projector onto $M$. □

4.6 Show that $P \in \mathscr{L}(H)$ is an orthogonal projector if and only if $(Px, x) = \|Px\|^2$ for all $x \in H$.

*Proof.* `Only if': Suppose that $P$ is a projector, then $x = y + z$ with $y \in R(P)$ and $z \in R(P)^\perp$. Then $(Px, x) = (y, y + z) = (y, y) + (y, z) = (y, y) = \|Px\|^2$.

`If': From Proposition 4.4.5 (1), we know that $P$ is symmetric, hence $P^2 - P$ is symmetric. Notice that $(P^2x - Px, x) = (P^2x, x) - (Px, x) = (Px, Px) - (Px, x) = 0$, it follows from Proposition 4.4.5 (5) that $P^2 - P = 0$. Hence $P$ is an orthogonal projector by the previous problem. □

4.7 Let $A \in \mathscr{L}(H)$, it is called positive operator if $(Ax, x) \geq 0$ for all $x \in H$. Show that

(1) All positive operators are symmetric;

(2) The eigenvalues of a positive operators are non-negative.

*Proof.* (1) Proposition 4.4.5(1).

(2) From Proposition 4.4.5(2), we need only to consider $\lambda I - A$ for real $\lambda$. Notice that for negative $\lambda$ it holds that

$$\|(\lambda I - A)x\|^2 = ((\lambda I - A)x, (\lambda I - A)x) = \lambda^2\|x\|^2 - 2\lambda(Ax, x) + \|Ax\|^2 \geq \lambda^2\|x\|^2$$

and thus $\lambda I - A$ is injective for negative $\lambda$. Hence $\sigma(A) \subset [0, \infty)$. □

4.8 Let $L$ and $M$ be two closed linear subspace of $H$. Show that $L \subseteq M$ iff $P_M - P_L$ is positive.

*Proof.* `Only if': Suppose that $x = x_L + y_L = x_M + y_M$ where $x_A \in A$ and $y_A \in A^\perp$. Decompose $x_M$ as $x_M = u + v$, where $u \in L$ and $v \in L^\perp$. Notice that $y_M \perp L$ hence $x = u + (v + y_M)$ is an orthogonal decomposition along $L$, and from the uniqueness of decomposition it must hold that $x_L = u$. Since $L \subseteq M$, it holds that $x_M - x_L \in M$ and thus $(P_M x - P_L x, x) = (x_M - x_L, x_M + y_M) = (x_M - x_L, x_M) = (v, u + v) = (v, v) \geq 0$.

`If': Suppose that $x \in L$ then $P_L x = x$. Suppose that $x = x_M + y_M$ where $x_M \in M$ and $y_M \in M^\perp$. Then $0 \leq (P_M x - P_L x, x) = (P_M x - x, x) = (x_M - x, x) = -(y_M, x_M + y_M) = -(y_M + y_M) \leq 0$, hence $y_M = 0$ and $x = x_M$, hence $x \in M$, and $L \subseteq M$. □

4.9 Let $(a_{ij})$ satisfy $\sum_{i,j=1}^\infty |a_{ij}|^2 < \infty$. Define in $\ell^2$

$$A : x = \{x_1, x_2, \dots\} \mapsto y = \{y_1, y_2, \dots\}$$

where $y_i = \sum_{j=1}^\infty a_{ij} x_j$. Show that

(1) $A$ is compact;

(2) If $a_{ij} = \overline{a_{ji}}$ then $A$ is a symmetric compact operator.

*Proof.* (1) Using Cauchy-Schwarz inequality it is easy to verify that $A \in \mathscr{L}(\ell^2)$. Define $A_N \in L(\ell^2)$ as

$$A : x = \{x_1, x_2, \dots\} \mapsto y = \{y_1, \dots, y_N, 0, \dots\}$$

then $A_N$ is a finite-rank operator and thus is compact. And it follows from Cauchy-Schwarz inequality that $\|A_N - A\| \to 0$, hence $A$ is compact.

(2) Suppose that $z = (z_1, z_2, \dots)$. It is not hard to show that

$$\left| \sum_{i=1}^N \sum_{j=1}^N a_{ij} x_j \overline{z_i} - \sum_{i=1}^\infty \sum_{i=1}^\infty a_{ij} x_j \overline{z_i} \right|^2 \to 0$$

as $N \to \infty$ using Cauchy-Schwarz inequality. Also,

$$\sum_{i=1}^N \sum_{j=1}^N a_{ij} x_i = \sum_{i=j}^N \sum_{j=i}^N \overline{a_{ji} z_i} x_j \to \sum_{j=1}^\infty \sum_{i=1}^\infty x_j \overline{a_{ij} z_i},$$

which can be proved in the same way. We have established $\langle Ax, z \rangle = \langle x, Az \rangle$, hence $A$ is symmetric. □

4.10 Let $A$ be a symmetric operator on $H$ and there exists an orthonormal basis in which every vector is an eigenvector of $A$. Suppose that

(1) $\dim N(\lambda I - A) < \infty$ ($\forall \lambda \in \sigma_p(A) \setminus \{0\}$)

(2) For any $\epsilon > 0$, the set $\sigma_p(A) \setminus [-\epsilon, \epsilon]$ is finite.

Show that $A$ is a compact operator on $H$.

*Proof.* From the two assumptions we know that $A$ has countably many eigenvalues (including the multiplicity). List them in the decreasing order of absolute value (with multiplicity) as $|\lambda_1| \geq |\lambda_2| \geq \cdots$. Since $A$ is symmetric, we know that the basis contains a basis of $N(\lambda I - A)$ for all eigenvalue $\lambda \neq 0$. Then $Ax = \sum_{i=1}^\infty (x, e_i) e_i$ (no need to consider the eigenvectors associated with eigenvalue 0), where $e_i$ is an eigenvector associated with $\lambda_n$. Define $A_N = \sum_{i=1}^N \lambda_n (x, e_i) e_i$ then $A_N$ is of finite rank and thus compact. By the Remark 1 of Theorem 4.4.7, $\|A - A_N\| \to 0$ and thus $A$ is compact. □